Processus stochastiques matriciels, systèmes de racines et probabilités non commutatives

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CHAPITRE 1

Matrices aléatoires : différents aspects et applications

1. Historique : version statique et indice de Dyson

Une réference de base et bien détaillée vers laquelle nous dirigeons les lecteurs est le livre de Mehta ([**88**]). Pour un point de vue algébrique ainsi que diverses applications en physique, on renvoie au manuscrit de Caselle et Magnea ([**26**]). D'autres références utiles seront fournies dans le cours du texte.

1.1. Motivations physiques. La physique, une source remarquable de problé mes, a été depuis plus d'un demi-siècle un terrain favorable où les matrices aléatoires jouent un rôle majeur. D'un point de vue historique ([**4**], [**18**]), l'histoire commence avec Wigner en 1955 et ensuite Dyson, même si les premiers travaux remontent à Wishart en 1928 dans l'étude de la dynamique des populations ([**113**]). Wigner a introduit les matrices alétoires afin de décrire les niveaux d'énergie d'un système complexe tel que le noyau atomique de l'isotope 239 de l'Uranium. En effet, leurs valeurs et vecteurs propres permettent d'approcher respectivement le spectre discret et les fonctions propres d'un Hamiltonien $\mathcal{H}$ opérant sur un espace de Hilbert de dimension infinie. Ce dernier satisfait l'équation de Schrödinger $\mathcal{H}\Psi = E\Psi$ où E, Ψ désignent l'énergie et la fonction propre correspondante. Ainsi, l'approximation par des matrices aléatoires carrées, justifiée par les simulations numériques (tel que la spectroscopie), nous ramène à étudier un problème de valeurs propres dans un espace de dimension grande, mais finie.
Wigner a commencé avec des matrices réelles symétriques formées par des variables aléatoires de Bernoulli symétriques et indépendantes. Il a montré la convergence faible de l'espérance de la mesure empirique des valeurs propres (aléatoire) vers la loi du demi-cercle (ou loi de Wigner). Par contre, il a signalé que son choix repose essentiellement sur les symétries que présentent le système et non pas sur la loi de chaque coefficient. Celà a été affirmé dans son papier en 1958 où son résultat couvrait une classe plus large de matrices réelles symétriques à coefficients indépendants ayant des lois symétriques, des moments d'ordre 2 finis et d'ordre 1 uniformément bornés. Plus exactement, si A est une matrice de Wigner de taille $m \times m$ et si $(\lambda_1, \ldots, \lambda_m)$ désignent ses valeurs propres, alors :

$$\mathbb{E}(F_m(x)) := \mathbb{E}\left\{\frac{1}{m}\sum_{k=1}^{m} \mathbf{1}_{\{\lambda_k \leq x\sqrt{m}\}}\right\} \longrightarrow F(x),\, m \to \infty$$

où F est la fonction de répartition de la loi de Wigner. Améliorant ce résultat, Arnold a obtenu la convergence en probabilité et presque sûre pour des matrices complexes hermitiennes lorsque les coefficients diagonaux sont iid et les hors diagonaux le sont aussi, toutes les variables étant indépendantes. Dans un premier temps, la convergence en probabilité nécessite une variance finie pour les éléments de la diagonale et un moment d'ordre 4 fini ainsi qu'une espérance nulle pour les autres coefficients. Dans un second temps, il a éliminé l'hypothèse sur les éléments diagonaux ainsi que celle de l'espérance nulle et a affaibli les autres pour se restreindre à des moments finis d'ordre au plus 4. Finalement, le résultat est maintenu seulement en faisant une hypothèse de variance finie. Plus tard, en 1988, Bai et Yin ont donné les conditions nécessaires et suffisantes pour la convergence presque sûre de la plus grande valeur propre vers la borne supérieure du support de la loi du demi cercle, la vitesse étant de $\sqrt{m}$. Il y a un résultat analogue pour la plus petite valeur propre et la borne inférieure du support. Par rapport aux travaux d'Arnold, la finitude de la variance des éléments diagonaux est requise. Une référence complète de ces faits avec des preuves à la fois détaillées et commentées est [**4**]. Bien evidemment, il est trés naturel d'essayer d'établir un TCL ainsi qu'un PGD pour la plus grande valeur propre normalisée par $1/\sqrt{m}$. Dans le cas Gaussien, le premier a fait l'objet des travaux de Tracy et Widom ([**109**], [**110**]), la vitesse étant de $m^{2/3}$. Le second se trouve dans les papiers de Ben Arous, Dembo et Guionnet ([**10**]). L'universalité de la loi de Tracy-Widom pour des matrices à coefficients iid ayant des lois symétriques a été établie par Soshnikov : ceci requiert une hypothèse de croissance sur les moments pairs appelés dans ce cas moments sous-Gaussiens. Il y a aussi un PGD pour la mesure spectrale qui fait intervenir une fonction de taux dont le minimum est atteint par la loi de Wigner ([**9**]).

Revenons à la remarque pertinente de Wigner concernant les symétries : celles-ci se résument en l'invariance par la conjugaison d'un certain groupe, ainsi qu'une propriété connue chez les physiciens sous le nom d'invariance par retournement du temps. En quelques mots, l'Hamiltonien vérifie une relation de commutation avec un opérateur dit "anti-unitaire". La classification a été faîte ensuite par Dyson qui a montré que le groupe d'invariance est soit le groupe orthogonal, unitaire ou symplectique et que les matrices sont respectivement réelles symétriques, complexes hermitiennes et hermitiennes auto-duales de taille paire dont les éléments sont des quaternions. Deux modèles auxquels Dyson s'est intéressé portent les noms de *Gaussien* et *circulaire*. Le premier requiert, en plus de l'invariance, l'indépendance des coefficients de la matrice. Les familles correspondantes sont notées GOE (GUE, GSE) comme abbréviations de : "Gaussian orthogonal ensemble" (unitary, symplectic). L'indépendance entraîne que la densité s'écrit sous la forme :

$$p(dA) = Ce^{-a\,\mathrm{tr}(A-cI)^2}dA, \quad a > 0,\, c \in \mathbb{R}$$

où dA est la mesure de Lebesgue et C est une constante positive ne dépendant que de la taille de la matrice. Dans ce cas, $p(dA)$ est invariante par translation et on peut se ramener à :

$$p(dA) = C(a)e^{-\operatorname{tr}(A^2/2)}dA. \tag{1}$$

La matrice peut être alors réalisée à partir de lois normales indépendantes. Par exemple, une matrice $\in$ GUE est donnée par :

$$A_{ij} = \begin{cases} \mathcal{N}^1_{ii}(0,1) & \text{si} \quad i = j \\ \mathcal{N}^1_{ij}(0,1/2) + \sqrt{-1}\,\mathcal{N}^2_{ij}(0,1/2) & \text{si} \quad i < j \end{cases}$$

où $(\mathcal{N}^1_{ij})_{i,j}$, $(\mathcal{N}^2_{ij})_{i,j}$ sont toutes indépendantes. Pour l'ensemble circulaire, on a les COE, CUE et CSE qui correspondent à des matrices unitaires symétriques, unitaires et unitaires à éléments quaternions. Elles sont choisies suivant la mesure de Haar normalisée (une telle mesure est finie puisque le groupe des matrices unitaires est compact).

1.2. Les matrices triangulaires : ensembles chiraux. Dans ce qui précède, nous n'avons cité que des matrices carrées. Ceci n'exclut évidemment pas les matrices rectangulaires de contribuer à la modélisation de certains phénomènes. Au contraire, comme il est déjà indiqué, les premiers travaux remontent à ([**113**]) qui a étudié à cette époque les matrices de covariance. Partant d'un échantillon de taille N et si l'on voulait étudier m caractères dans une population donnée, chaque individu i est représenté par un vecteur colonne E_i. On forme alors la matrice rectangulaire $E^T = (E_i^T)_{1\leq i\leq N}$. Posons $n := N-1$, alors un estimateur non biaisé de la matrice de covariance est :

$$S = \frac{1}{n}\sum_{i=1}^{N}(E_i - \overline{E})(E_i - \overline{E})^T := \frac{1}{n}A, \quad \overline{E} := \frac{1}{N}\sum_{i=1}^{N}E_i$$

Il est montré dans [**89**] que, dans le cas où E_i sont des vecteurs Gaussiens iid, A a la même loi que N^TN où N est une matrice $n\times m$ formée par des vecteurs lignes Gaussiens indépendants. Cette nouvelle matrice carrée réelle symétrique et positive est appelée *matrice de Wishart* $W(n,m)$. Elle est caractérisée par la matrice des espérances M et celle des covariances Σ de chaque vecteur (En effet, Σ est une matrice diagonale par blocs). Lorsque M est non-nulle, la loi de Wishart est dîte *décentrée*, M et Σ sont les paramètres de décentrage. Il est à noter que la somme de matrices de Wishart indépendantes $W(n_1,m)$, $W(n_2,m)$ est encore une matrice de Wishart (n_1+n_2,m) (additivité). Quand $m=1$, le résultat est bien connu et A suit la loi de Chi-deux à n degrés de liberté $\chi^2(n)$. L'analogue complexe hermitien de N^TN est appelé matrice de Wishart complexe. Une vaste littérature couvrant les moments et l'étude des valeurs propres de A se trouve dans les travaux de James, Letac, Massam, Graczyk, Muirhead et Chikuze ([**59**], [**60**], [**61**], [**31**], [**69**], [**89**]). Lorsque $\Sigma = I_n$ (tous les coefficients sont iid) et la matrice de Wishart est inversible (définie positive), la densité prend la forme :

$$p(dA) = C(n,m)\det(A)^{\beta(n-m+1)/2-1}e^{-\operatorname{tr}(A)/2}\mathbf{1}_{\{A>0\}}dA, \tag{2}$$

où $\beta = 1, 2, 4$ selon que la matrice multivariée de départ est réelle, complexe ou auto-duale quaternionique. Le paramètre β est connu sous le nom *d'indice de Dyson*. On verra plus loin qu'il apparaîtra dans l'expression du Jacobien résultant de la décomposition de la matrice en partie radiale (valeurs propres) et en partie angulaire (vecteurs propres). Il nous permettra également de caractériser chacun des trois ensembles. Suivant la terminologie de Dyson, une matrice ayant la densité (2) appartient à LOE ($\beta = 1$), LUE ($\beta = 2$) et LSE ($\beta = 4$) : "*Laguerre orthogonal, unitary, symplectic ensembles*". Cette appelation est relative à la présence des polynômes de Laguerre dans l'étude de ces modèles. D'autres applications apparaîssent aussi pour modéliser divers phénomènes physiques que nous ne discuterons pas ici et qui sont à l'origine du mot "chiral". Dans ces situations, l'indice de Dyson joue le rôle de l'inverse de l'énergie thermique kT où k est la constante de Boltzmann. Le choix entre orthogonal, unitaire et symplectique dépend comme d'habitude des symétries que présentent le système étudié. Les résultats asymptotiques analogues à ceux du cas des matrices de Wigner sont présents. Commençons par Marchenko et Pastur qui ont montré la convergence de la mesure spectrale, lorsque $m/n \to y \in (0, \infty)$, vers la loi portant leurs noms. Celle-ci peut éventuellement avoir une masse de Dirac en 0 dépendant de y, suivant que $n \geq m$ ou $n < m$. La surprise est que la mesure spectrale converge presque sûrement vers la loi de Wigner quand $m/n \to 0$. Dans cette direction, une LGN, un TCL et un PGD pour la plus grande valeur propre ont été établis.
Terminons ce paragraphe par un modèle classique étroitement lié aux matrices de Wishart : la matrice de Jacobi ou MANOVA. Etant donné deux matrices de Wishart indépendantes $W(n_1, m)$, $W(n_2, m)$ tel que leur somme $W(n_1 + n_2, m)$ est inversible, cette matrice notée $J(n_1, n_2, m)$ est définie par

$$[W(n_1, m) + W(n_2, m)]^{-1/2} W(n_1, m) [W(n_1, m) + W(n_2, m)]^{-1/2}$$

Suivant qu'on a des éléments du LOE, LUE et LSE, on obtient des éléments du JOE, JUE et JSE. Si $W(n_1, m)$, $W(n_2, m)$ sont inversibles, alors il en est de même pour J et $I_m - J$ et la densité de J s'écrit :
(3)

$$p(dA) = C(n_1, n_2, m) \det(A)^{\beta(n_1 - m + 1)/2 - 1} \det(I_m - A)^{\beta(n_2 - m + 1)/2 - 1} \mathbf{1}_{\{0 < A < I_m\}} dA$$

où I_m désigne la matrice identité.

1.3. Des matrices déformées. Motivés par des problèmes de la théorie de l'apprentissage, de l'analyse des données financières et du traitement du signal, les chercheurs se sont affrontés à des matrices *déformées*. La déformation sous-entend par exemple, la perturbation par une matrice de rang fini pour le cas GUE ou GOE, ou bien une légère modification des valeurs propres de $\Sigma = I_n$ d'une matrice du LUE ou LOE (Wishart non blanc). Le modèle est choisi de façon à ce que le régime global reste invariant, c'est à dire que la mesure spectrale limite est la même que dans le cas non déformé. Par contre, des changements au niveau de la convergence presque sûre et les fluctuations concernent la plus grande valeur propre ou bien même les premières plus grandes, dépendant du rang de la perturbation. Dans le cas GUE et lorsque on ajoute une matrice de rang 1 (bien normalisée), il est montré que la plus grande valeur propre pouvait sortir du support et que ses fluctuations pouvait changer de loi (Tracy-Widom) et (ou) de vitesse. Tout dépend de la plus grande valeur propre de la matrice déformante et de sa multiplicité. Des résultats similaires ont été établis pour le modèle du Wishart non blanc. Pour les grandes déviations, le cas rang fini $\neq 1$ reste encore ouvert. Pour plus de détails et de résultats sur l'universalité, nous renvoyons à la thèse de D. Féral et les références qui y sont ([**54**]). Le manuscrit [**4**] renferme d'autres modèles assez intéressants que nous recommandons vivement au lecteur. Terminons ce paragraphe par mentionner que les matrices aléatoires interviennent aussi en théorie des nombres, en file d'attente et en combinatoire ([**18**], [**43**]).

1.4. Densité des valeurs propres. L'ingrédient principal nous permettant de mener les calculs est la densité de la loi jointe des valeurs propres. Celle-ci peut être déduite de celle de la matrice soit par un calcul différentiel ([**89**]), soit moyennant la formule d'intégration de Weyl. Notons $\lambda = (\lambda_1, \dots, \lambda_m)$ les valeurs propres de la matrice carrée $m \times m$. Les densités (1), (2) et (3) se transforment en :

$$\begin{aligned}
p(\lambda) &= C(m)e^{-\sum_{i=1}^m \lambda_i^2/2}\prod_{i<j}|\lambda_i-\lambda_j|^\beta,\\
p(\lambda) &= C(n,m)e^{-\sum_{i=1}^m \lambda_i/2}\prod_{i=1}^m \lambda_i^{\beta(n-m+1)/2-1}\prod_{i<j}|\lambda_i-\lambda_j|^\beta \mathbf{1}_{\{\lambda>0\}}\\
p(\lambda) &= C(n_1,n_2,m)\prod_{i=1}^m \lambda_i^{\beta(n_1-m+1)/2-1}\prod_{i=1}^m(1-\lambda_i)^{\beta(n_2-m+1)/2-1}\prod_{i<j}|\lambda_i-\lambda_j|^\beta \mathbf{1}_{\{0<\lambda<1\}}
\end{aligned}$$

où $\beta = 1, 2, 4$. Par un changement de variable si nécessaire, $p(\lambda)$ peut se mettre sous la forme $C(m,\beta)e^{-\beta W(\lambda)}$ où W désigne l'énergie potentielle du système. Cette forme est connue sous le nom de " facteur de Boltzmann". Comme la température T est un paramètre positif, il en est de même pour β ($\beta = 1/kT$) et il n'y a pas de raison de faire la restriction $\beta = 1, 2, 4$. Ainsi, dans ce modèle, les valeurs propres se comportent comme des particules à la température T soumises au potentiel W. En mécanique statistique, ce sont des charges ponctuelles d'un gaz en équilibre thermodynamique : c'est un gaz de Coulomb. De plus, leurs interactions sont données par $|\lambda_i - \lambda_j|^\beta$. Si $\beta \to 0$, ($T \to \infty$), ce terme disparaît et les particules se meuvent indépendamment les unes des autres : le système est très chaud. Si $\beta \to \infty (T \to 0)$, le système est gelé et les particules sont immobilisées. Il est alors légitime de chercher des modèles matriciels à coefficients le moins correlés possibles, voire indépendantes, et dont les valeurs propres ont les densités ci dessus avec $\beta > 0$. Ce problème a été résolu pour les deux premiers cas (Gaussien,

Wishart) dans [**45**] et est par Killip et Nenciu ([**77**]) pour le cas circulaire et Jacobi. Les modèles mis en évidence sont triangulaires symétriques et portent les noms respectifs d'ensembles β-Hermite et β-Laguerre. Les variables constituant la matrice sont des lois normales, χ pour le premier ensemble, des lois de χ^2 pour le deuxième et finalement des produit de lois Beta pour le dernier, évidemment toutes dépendantes de β.
Retournons un instant à l'expression de W. Celle-ci se décompose en deux termes : un potentiel harmonique attirant chaque valeur propre indépendamment des autres et une répulsion électrostatique :

$$W(\lambda) = \sum_{i=1}^{m} V_1(\lambda_i) + \sum_{1\le i<j\le m} V_2(\lambda_i, \lambda_j)$$

Pour le premier cas, $p(\sqrt{\beta}\lambda)$ correspond à

$$W(\lambda) = \frac{1}{2}\sum_{i=1}^{m} \lambda_i^2 - \sum_{1\le i<j\le m} \log|\lambda_i - \lambda_j| \tag{4}$$

Dans le deuxième cas, le changement de variable $\lambda_i \mapsto \beta\lambda_i^2$ nous donne :

$$W(\lambda) = \frac{1}{2}\sum_{i=1}^{m} \lambda_i^2 - [(n-m+1) - \frac{1}{\beta}]\sum_{i=1}^{m} \log(\lambda_i) - \sum_{1\le i<j\le m} \log|\lambda_i^2 - \lambda_j^2| \tag{5}$$

Le dernier est soumis au changement $\lambda_i \mapsto \sin^2\lambda_i$. On peut aussi associer à cette énergie potentielle un Hamiltonien $\mathcal{H}$ admettant $e^{-\beta W/2}$ comme fonction propre associée à l'énergie minimale, disons $-E_0$. Elle porte le nom de fonction d'onde. $\mathcal{H}$ a pour expression :

$$\mathcal{H} - E_0 = -\sum_{i=1}^{m} \partial_i^2 - \frac{\beta}{2}\sum_{i=1}^{m}(\partial_i^2 W) + \frac{\beta^2}{4}\sum_{i=1}^{m}(\partial_i W)^2 \tag{6}$$

Ceci fait la connection avec la famille "Calogero-Moser-Sutherland" (CMS) de certains systèmes qui jouissent d'une propriété dîte "d'intégrabilité" rendant leurs études plus accessibles.
D'un point de vue algébrique, $p(\lambda)$ fait intervenir des termes du type $<\alpha,\lambda>$ où $<,>$ désigne le produit scalaire euclidien dans $\mathbb{R}^m$ et pour certains α appelées *racines*. Ces racines définissent des hyperplans orthogonaux par rapport auquels on exerce des réflexions. Un ensemble de ces racines qui est globalement invariant par ces réflexions constitue *le système de racines* et engendre un sous espace de $\mathbb{R}^m$. Ceci nous plonge dans un monde dans lequel interviennent des groupes de réflexions finis et de réflexions affines infinis (translations). De plus, la statistique d'ordre des valeurs propres est vue comme un élément d'un domaine convexe définie à partir des racines : *chambres et cellules de Weyl.* Par ailleurs, l'ensemble des matrices associées constitue un groupe de Lie $\mathbf{G}$ d'une certaine algèbre de Lie $\mathfrak{G}$ (appelée algèbre de Lie linéaire spéciale, [**25**]). Les valeurs propres sont alors vues comme la

partie radiale lors du passage des coordonnées locales du groupe aux coordonnées polaires.

2. La version dynamique : processus matriciels

Dans cette thèse, on s'intéressera principalement aux modèles dépendant du temps, tout en remplaçant les variables aléatoires par des processus stochastiques. On parle alors de processus matriciels. Ce qui change par rapport au cas statique est l'utilisation des outils probabilistes tels que le calcul stochastique, les semi-groupes (équations backward ou Fokker-Planck), les générateurs et les théorèmes limites afin de comprendre l'évolution de notre processus. Ceci est un point fort dans le sens où, par exemple, pour t fixé ($= 1$), en faisant partir le processus de 0, on retrouve le cas statique. De plus, des EDS pour les processus matriciels analogues aux cas univariés ainsi que pour leurs valeurs propres sont écrites. Au niveau matriciel, elles présentent une sorte de symétrie due à la non commutativité. Cependant, certains outils classiques ne seront plus valables ce qui rend la preuve de l'existence et de l'unicité des solutions assez compliquée. Au niveau spectral, les EDS peuvent présenter (au moins dans les cas qu'on connaît) une dérive singulière qui explose lorsque deux particules se touchent ou une particule touche un "mur". Ceci justifie bien l'interaction entre les particules et nous oblige à prendre soin du premier temps de collision ainsi que de la condition initiale. Cette dérive montre aussi la corrélation entres les différentes valeurs propres. Cependant, et seulement dans le cas hermitien complexe, le processus des valeurs propres est une **h-transformée** (pour une fonction h bien choisie, [**101**]) d'un processus à composantes indépendantes tué s'il y a collisions. Ceci se voit explicitement au niveau des générateurs et nous permet d'écrire, moyennant la formule de Karlin-McGregor ([**72**]), la densité du semi groupe sous forme de déterminant ([**78**]). Citons quelques exemples pour fixer les idées : le *Mouvement brownien hermitien* de Dyson ([**48**]). Le processus est défini par :

$$X_{ij}(t) = \begin{cases} B_{ii}(t) & \text{si} \quad 1 \leq i = j \leq m \\ \left(\dfrac{B^1_{ij}(t) + \sqrt{-1} B^2_{ij}(t)}{\sqrt{2}} \right) & \text{si} \quad 1 \leq i < j \leq m \end{cases}$$

où $(B_{ii})_i$, $(B^1_{ij})_{i,j}$, $(B^2_{ij})_{i,j}$ sont des familles indépendantes de mouvements browniens indépendants. La densité s'écrit alors :

$$p_t(dX) = C(t,m) e^{-\operatorname{tr} X^2/2t} dX$$

et donc celle des valeurs propres est

$$p_t(\lambda) = C(m,t) e^{-\sum_{i=1}^m \lambda_i^2/2t} \prod_{i<j} |\lambda_i - \lambda_j|^2$$

Pour $t=1$, on retrouve la densité du GUE. Si on part de $\lambda_1(0) > \cdots > \lambda_m(0)$, alors les valeurs propres ne se touchent plus presque sûrement et

$$d\lambda_i(t) = d\nu_i(t) + \sum_{j\neq i} \frac{dt}{\lambda_i(t)-\lambda_j(t)}, \quad 1 \leq i \leq m, \tag{7}$$

où $(\nu_i)_{1\leq i\leq m}$ est un mouvement brownien m-dimensionnel. La fonction de Vandermonde

$$V(x) = \prod_{i<j}(x_i - x_j), \quad x_1 > \cdots > x_m$$

vérifie $\triangle V(x) = 0$ (harmonique) et λ est la V-transformée d'un processus formé par m mouvements browniens indépendants tué lorsque deux composantes s'intersectent ([**78**]). Ceci permet d'écrire la densité du semi-groupe du processus $(\lambda_i(t),\, 1 \leq i \leq m)_{t\geq 0}$ sous la forme :

$$p_t(\eta,\lambda) = \frac{V(\lambda)}{V(\eta)} \det\left[(1/\sqrt{2\pi t})e^{-(\lambda_i-\eta_j)^2/2t}\right]_{i,j}, \quad \eta = (\lambda_i(0),\, 1\leq i\leq m).$$

Dans la littérature ([**30**] par exemple), on trouve la version stationnaire de ce processus (Ornstein-Ulhenbeck) donnée par une variance $1-e^{-2t}$. Par analogie avec les GOE, GUE et GSE, on peut parler des mouvements browniens symétrique et symplectique. Les EDS correspondantes s'écrivent pour tout $t \geq 0$ ([**28**])

$$d\lambda_i(t) = d\nu_i(t) + \frac{\beta}{2}\sum_{j\neq i} \frac{dt}{\lambda_i(t)-\lambda_j(t)},\, \beta = 1,\, 4. \quad 1 \leq i \leq m \tag{8}$$

et l'indice de Dyson apparait encore une nouvelle fois. A $t=1$, on a des réalisations de matrices du GOE, GUE et GSE.
Plus tard, M. F. Bru ([**19**]) étudie le processus de Wishart défini par $X_t = B_t^T B_t$ où B est une matrice brownienne rectangulaire $n\times m$. Le processus ainsi obtenu est positif, de taille $m \times m$ et vérifie l'EDS :

$$dX_t = B_t^T dB_t + dB_t^T B_t + nI_m dt,\; X_0 = B_0^T B_0. \tag{9}$$

n est sa dimension et on note $W(n,m,X_0)$. Pour $m=1$, c'est un carré de Bessel ([**101**]) de dimension n. A t fixé, la loi de X_t est la *loi de Wishart décentrée* de paramètres $M = X_0$ et $\Sigma = tI_m$ ([**69**], [**89**]). Si de plus, $X_0 = 0$, X_1 est une matrice du LOE. Il est à noter que le processus jouit d'une propriété d'additivité, c'est à dire que la somme de deux processus de Wishart indépendants de dimensions n_1, n_2 et de taille $m\times m$ est aussi un processus de Wishart de dimension n_1+n_2. Malheureusement, la loi n'est pas indéfiniment divisibleà cause de la forme de l'ensemble de Gindikin : en effet, pour $n \geq m$, X_t est presque sûrement définie positive pour tout $t>0$ si X_0 l'est, et donc inversible. (9) peut donc s'écrire :

$$dX_t = \sqrt{X_t}dN_t + N_t^T\sqrt{X_t} + nI_m dt$$

où N est une MB $m\times m$. Celà suggère de définir le processus de Wishart d'indice $\delta > 0$ comme solution de cette EDS, lorsqu'elle existe, en remplaçant n par

$\delta > 0$. D'aprés ([**19**]), le $W(\delta, m, X_0)$ existe pour δ dans l'ensemble de Gindikin $\{1, \ldots, m-1\} \cup]m-1, \infty[$. Les valeurs propres du processus de Wishart satisfont $\forall t < \tau := \inf\{s,\, \lambda_i(s) = \lambda_j(s) \,\text{pour un couple}\, (i,j)\}$ ([**20**]) :

$$d\lambda_i(t) = 2\sqrt{\lambda_i(t)}d\nu_i(t) + \left[\delta + \sum_{j \neq i} \frac{\lambda_i(t) + \lambda_j(t)}{\lambda_i(t) - \lambda_j(t)}\right] dt, \quad 1 \leq i \leq m, \tag{10}$$

où $(\nu_i)_i$ est un MB de dimension m et $\lambda_1(0) > \cdots > \lambda_m(0)$. On voit bien qu'il y a un terme qui correspond à un carré de Bessel de dimension δ et un terme singulier montrant les interactions. On montre que dans ce cas, le temps de collision est infini presque sûrement. D'autres processus matriciels tels que l'O-U et son carré figurent dans [**19**]. Dans le but d'étendre les propriétés connues en dimension 1, Donati et al. ont établi des relations d'absolue-continuité, l'inversion du temps $(t \mapsto 1/t)$, la loi de Hartman-Watson généralisée et la queue de répartition de $T_0 := \inf\{t > 0,\, \det(X_t) = 0\}$ ([**40**]). Il est aussi à mentionner que le calcul des moments de X_t a fait l'objet de [**61**].

Remarque. Le processus de Wishart a été considéré par les physiciens ([**30**]) mais nous renvoyons son introduction à M. F. Bru pour son étude probabiliste très détaillée.

La version complexe, appelée mystérieusement *processus de Laguerre*, est apparue dans [**78**] où les auteurs se sont intéressés à ses valeurs propres et dans [**74**]. L'analogue de (10) est (le temps de collision est aussi infini p.s.)

$$d\lambda_i(t) = 2\sqrt{\lambda_i(t)}d\nu_i(t) + 2\left[\delta + \sum_{j \neq i} \frac{\lambda_i(t) + \lambda_j(t)}{\lambda_i(t) - \lambda_j(t)}\right] dt, \quad 1 \leq i \leq m, \tag{11}$$

La fonction V est encore harmonique mais cette fois pour le générateur de m carrés de Bessel indépendants et la caractérisation de V-processus a lieu aussi. Par contre, nous n'avons pas rencontré d'étude analogue à celle du cas réel, chose qui nous a encouragé à la faire dans cette thèse. Mais la vraie raison était la souplesse de la structure complexe qui dans un premier temps se manifeste à travers les valeurs propres et dans un second temps rend les calculs plus accessibles et donc les résultats plus fins. Ceci est valable pour d'autres processus, le processus de Jacobi matriciel par exemple ([**43**]). Commençons par introduire la cas univarié : $m = 1$. Ce processus est l'unique solution forte de ([**49**])

$$dJ_t = 2\sqrt{J_t(1-J_t)}dB_t + (p - (p+q)J_t)dt, \quad p,\, q > 0, \quad J_0 \in [0,1].$$

Le cas matriciel réel est défini comme suit : soient $m,\, p,\, d \in \mathbb{N}^\star$ tels que $m,\, p \leq d$ et $Y(d)$ une matrice brownienne orthogonale $d \times d$. Notons $X(m,p)$ le coin supérieur gauche $m \times p$ de $Y(d)$. Alors $J_t := X_t(m,p)X_t^T(m,p)$ et est noté $J(p,q)$ avec $q = d - p$. L'EDS s'écrit alors :

$$dJ_t = \sqrt{J_t}dB_t\sqrt{I_m - J_t} + \sqrt{I_m - J_t}dB_t^T\sqrt{J_t} + (pI_m - (p+q)J_t)dt$$

où B est une matrice brownienne carrée et $0 \leq J_t \leq I_m$. Ensuite, comme d'habitude, on étend cette définition au cas des paramètres non entiers en étudiant l'existence d'une solution pour cette EDS. Notons bien que les coefficients diagonaux sont des processus de Jacobi univariés et que $I_m - J$ est un processus de Jacobi $J(q,p)$. La version complexe est définie de la même manière en prenant une matrice unitaire $Y(d)$ et l'EDS est satisfaite avec un indice de Dyson $\beta = 2$ devant le terme à variation finie. Ceci se transmet aux valeurs propres et on a :
(12)

$$d\lambda_i(t) = 2\sqrt{\lambda_i(1-\lambda_i)(t)}d\nu_i(t) + \beta\left[p - (p+q)\lambda_i(t) + \sum_{j\neq i}\frac{\lambda_i(1-\lambda_j)(t) + \lambda_j(1-\lambda_i)(t)}{\lambda_i(t) - \lambda_j(t)}\right]dt$$

pour $\beta = 1, 2$ et on voit bien la somme d'un processus de Jacobi univarié et d'un terme d'interactions. Les lecteurs peuvent rencontrer (dans des travaux d'analystes par exemple) un processus de Jacobi dont l'espace d'état est l'intervalle $[-1,1]$ au lieu de l'intervalle $[0,1]$ avec un opérateur de Jacobi égal au double du générateur ([**114**]). La transformation $x \mapsto 2x-1$ fait passer d'un processus à l'autre et un changement de temps déterministe de l'opérateur de Jacobi au générateur. Ce qui rend l'étude de ce processus assez difficile et parfois ennuyeuse est l'expression du semi groupe ([**100**], [**114**]). En dimension 1, on connaît une décomposition spectrale avec des polynômes de Jacobi qui sont à la fois les fonctions propres du générateur associé au processus et une base Hilbertienne de l'espace $L^2([-1,1])$ muni de la loi Beta comme mesure d'orthogonalité. Plus simplement, si $P_n^{\alpha,\beta}$ est le polynôme de Jacobi de degré $n \geq 0$ et de paramètres $\alpha, \beta > -1$, alors la densité du semi groupe s'écrit pour x, $y \in [0,1]$:

$$(13) \qquad p_t(x,y) = \sum_{n\geq 0} C_{n,\alpha,\beta} e^{-r_n t} P_n^{\alpha,\beta}(2x-1) P_n^{\alpha,\beta}(2y-1) y^{\beta}(1-y)^{\alpha}$$

où $r_n = n(n+\alpha+\beta+1)$ est la suite des valeurs propres. On ne connaît pas de formule fermée pour cette expression comme c'est le cas pour le mouvement Brownien réel (ou bien O-U) et les carrés de Bessel (ou carrés d'O-U), où les $P_n^{\alpha,\beta}$ sont remplacés par des polynômes de d'Hermite et de Laguerre respectivement. Ce qui fait la différence est que dans ces deux derniers cas, la suite r_n n'est plus quadratique en n et vaut n. Les formules ainsi obtenues sont du type *Mehler*.
Rappelons à titre d'information un modèle de valeurs propres qui n'est pas homogène en temps (ce qui n'est pas le cas des autres modèles cités ici). La densité de son semi groupe $p_{t,s}(\eta, \lambda)$ n'est pas une fonction de $t-s$. Ce modèle est dû à Katori et Tanemura et est limite normalisée de marches alèatoires conditionnées à ne pas se toucher jusqu'à un instant $T > 0$ fini (appelées marches"vicieuses", [**74**]). Dans un premier temps, les auteurs montrent que lorsqu'on part de 0 à $t = 0$, la loi du vecteur aléatoire à l'instant T est absolument continue par rapport à celle du vecteur des valeurs propres du MB hermitien au même instant. Il est montré aussi (et c'est intuitif) que ce processus converge vers le processus des

valeurs propres de Dyson quand $T \to \infty$. Dans un second temps, un processus matriciel hermitien formé de deux familles indépendantes de mouvements browniens indépendants et de ponts browniens de longueur T indépendants est fourni et correspond à ce système de particules. Un aspect intéressant de ce processus se manifeste par l'EDS qui se décompose en un premier terme qui n'est autre que (7) et d'un autre terme faisant intervenir la densité d'une matrice du GOE. Il en résulte qu'à t fixé, ce modèle est une sorte d'interpolation entre le GUE et le GOE ([**75**]).

3. Principaux résultats

3.1. Résumé du chapitre 3 : Laguerre processes and Generalized Hartman-Watson Law. Etant intitulé ainsi, ce travail fait l'objet d'étude de la version complexe du processus de Wishart connu sous le nom de processus de Laguerre. Ce processus est défini par $X_t = B_t^\star B_t$ où B est une matrice brownienne rectangulaire $n \times m$. le processus est de *dimension* n, de *taille* m et part de $X_0 = B_0^\star B_0$. Pour t fixé, la loi de la matrice X_t est la loi de Wishart complexe décentrée de paramètres $M = X_0$ et $\Sigma = 2tI_m$ ([**69**]). Ce travail a pour but de donner des résultats plus précis que ceux obtenus dans le cas réel concernant certaines lois de variables alétoires définies à partir de ces processus. En effet, comme on verra plus loin, les fonctions spéciales multivariées qui interviennent dans le cas complexe possèdent des représentations déterminantales, propriété qu'on ne trouve pas dans le cas réel.
Dans un premier temps, on se limite au cas des dimensions entières : on détermine l'expression du générateur et on établit d'une façon détaillée l'EDS vérifiée par les valeurs propres pour $n \geq m$. Ceci nous sera utile pour montrer que le processus reste presque sûrement défini positif si X_0 est définie positive. En plus, si les valeurs propres sont distinctes à $t = 0$, alors elles resteront presque sûrement distinctes pour tout t et tout $n \geq m - 1$. Ensuite, on utilise le générateur pour calculer la transformée de Laplace et par conséquent établir l'expression du semi groupe du processus. On en déduit et on retrouve la densité du semi groupe des valeurs propres moyennant la formule de Weyl et une des représentations déterminantales. Dans un second temps, on établit une EDS pour le processus X du type “carré de Bessel”, plus précisément X est l'unique solution forte de

$$dX_t = \sqrt{X_t} dN_t + dN_t^\star \sqrt{X_t} + 2\delta I_m dt$$

pour tout réel $\delta \geq m$, où N est une matrice brownienne carrée $m \times m$ et X_0 est définie positive. Pour X_0 positive et $\delta > m-1$, cette EDS a une unique solution en loi. δ est encore la dimension de X. Ceci étant fait, on étend les résultats précédents au cas des dimensions non entières et on établit les relations d'absolue-continuité. A partir de là, on écrit la queue de répartition de la variable $T_0 := \inf\{t, \det(X_t) = 0\}$ et on définit *la loi de Hartman-Watson généralisée* d'une manière analogue à celle du cas réel. En effet, cette dernière est définie par sa transformée de Laplace. Enfin, et pour $m = 2$, on utilise une autre représentation déterminantale pour expliciter la densité de la loi de chacune de ces variables aléatoires.

3.2. Résumé du chapitre 4 : Radial Dunkl Processes : Existence and uniqueness, Hitting time, Beta Processes and Random Matrices. Il est déjà mentionné dans l'introduction que l'indice de Dyson β est vu comme l'inverse de la température T. A partir de cette observation, il est tout a fait légitime de considérer toutes les valeurs positives que pourra prendre ce paramètre, même si cela va nous éloigner des cas "algébriques" $\beta = 1, 2, 4$. Pour étudier les EDS (7), (8), (10), (12) avec des indices $\beta > 0$, conduisant à ce qu'on appellera β-**processus**, on a eu recours au **processus de Dunkl radial**. Pour introduire ce processus, quelques définitions sont requises. Soit $(V, <>)$ un espace Euclidien de dimension m. On appelle **système de racines** et on note R, un ensemble de vecteurs non nuls qui engendrent V et qui est globalement invariant par l'ensemble des réflexions

$$\sigma_\alpha(x) := x - 2\frac{< \alpha, x >}{< \alpha, \alpha >}\alpha, \quad \alpha \in R,\, x \in V.$$

Il est *réduit* si $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$. On définit une relation d'ordre dans R et on parle de racines positives et négatives. Ceci peut se faire de plusieurs façons. Par exemple, on peut se donner un vecteur $u \in V$ et dire qu'une racine α est positive ssi $< u, \alpha > \geq 0$. On peut aussi trouver une base telle que toute racine est une combinaison linéaire positive ou négative des éléments de la base. Une telle base est appelée **système simple**, noté Δ, et ses éléments sont les **racines simples**. De plus, il partitionne R en deux : l'ensemble des racines positives est appelé **système positif** et est noté R_+. Le groupe engendré par toutes les réflexions est appelé groupe de réflexions ou bien parfois groupe de Weyl lorsqu'il stabilise un réseau. Il est fini, agit sur R et est noté W . La **chambre de Weyl positive** C est l'ensemble des vecteurs x de V tels que $< \alpha, x >> 0$ pour tout $\alpha \in \Delta$. Son adhérence est un cône convexe et tout de point de V est conjugué à un et un seul point de $\overline{C}$.

Le processus de Dunkl radial est un processus de Markov à trajectoires continues, à valeurs dans $\overline{C}$ et de générateur infinitésimal donné par :

$$(\mathscr{L}f)(x) = \frac{1}{2}(\Delta f)(x) + \sum_{\alpha \in R_+} k(\alpha)\frac{< \alpha, (\nabla f)(x) >}{< \alpha, x >}$$

où k est une fonction positive constante sur chaque orbite de R connue sous le nom de **fonction de multiplicité**. Nous avons commencé par montrer que l'EDS associé à ce générateur admet une unique solution forte pourvu que k soit strictement positive. On connaît d'autres preuves de ce résultat et qui sont analytiques basées sur des problèmes de martingales ([**33**], [**99**]). Celle dont on dispose est plutôt algéberique et utilise un résultat de Cépa et Lépingle sur l'existence et l'unicité d'une solution forte à une EDS avec un drift singulier ([**28**]). Ensuite, on s'intéressera au premier temps d'atteinte de ∂C. Il est redémontré via calcul stochastique que si $0 < k(\alpha) < 1/2$ pour une de ces racines, alors ce temps d'atteinte est fini presque sûrement (voir [**33**] pour la preuve utilisant essentiellement les martingales locales). La liaison avec les β-processus, entre autres les processus des valeurs propres déjà cités, est par la suite éclaircie, chose qui nous a permis d'expliciter les densités des semi groupes associés et de renforcer, au niveau des valeurs propres, des résultats déjà établis en étudiant les processus matriciels correspondants ($\beta = 1, 2, 4$). Il est important de signaler que

(1) les fonctions spéciales qui apparaîssent deviennent plus difficiles à manipuler dès qu'on s'écarte des cas $\beta = 1, 2, 4$. Ceci s'explique par le fait que dans ces trois cas, les β-processus sont des processus de valeurs propres de certains processus matriciels symétriques, hermitien ou hermitien auto dual. Par conséquent, les groupes orthogonal, unitaire et symplectique permettent le passage de $\mathbb{R}^m$ vers un espace de matrices, fait qui facilite parfois les calculs.

(2) le β-processus associé à l'EDS (12) sort du cadre du processus de Dunkl radial. Cette liaison a été déjà établie par Beerends et Opdam dans [**8**] où les auteurs identifient les fonctions hypergéométriques de Gauss avec des fonctions hypergéométriques associées au système de racines non réduit BC_m. Le cas ultrasphérique $p = q$ se contente du système réduit C_m. La même chose est faîte pour les polynômes de Jack et de Jacobi multivariés et ceux de Jacobi associés au systm̀e A_{m-1} et BC_m respectivement. De plus, on est en relation directe avec un groupe de réflexions affines engendré par

$$\tilde{\sigma}_{\alpha,k}(x) = x - 2(< \alpha, x > -k)\frac{\alpha}{< \alpha, \alpha >}, \qquad k \in \mathbb{Z},$$

et le processus vit dans une **cellule de Weyl**.

La fin du papier est consacrée à l'étude du temps d'atteinte de la frontière de la cellule de Weyl, à létude du mouvement brownien dans la cellule de Weyl et à l'écriture de la densité du semi groupe du processus β-Jacobi. Celle-ci nous permettra de répondre à quelques questions qui étaient ouvertes lors de l'étude du cas réel $\beta = 1$ ([**43**]).

3.3. Résumé du Chapitre 5. Nous avons vu au cours du chapitre précédent que le processus de Dunkl radial permettait de mieux comprendre un système formé par m particules en interactions et en particulier, pour $\beta = 1, 2, 4$, le processus des valeurs propres de certains processus matriciels. Cependant, les résultats obtenus restent valables au niveau spectral et on aimerait bien construire des processus matriciels qui soient "les plus simples possible" et ayant pour valeurs propres les β-processus. Par "plus simple", on sous-entend un modèle symétrique ou hermitien à coefficients le moins corrélés possible, voire indépendants. Considérons par exemple l'EDS du processus β-Dyson :

$$d\lambda_i(t) = d\nu_i(t) + \frac{\beta}{2}\sum_{j\neq i}\frac{dt}{\lambda_i(t)-\lambda_j(t)}, \qquad \beta > 0.$$

avec la condition initiale $\lambda_1(0) > \cdots > \lambda_m(0)$ et des ν_i indépendants. Nous avons abouti, pour $0 < \beta \leq 2$, à un modèle hermitien qui ressemble au mouvement brownien de Dyson défini par

$$X_{ij}(t) = \begin{cases} B_{ii}(t) & \text{si} \quad i = j \\ \sqrt{\dfrac{\beta}{2}}\left(\dfrac{B^1_{ij}(t) + \sqrt{-1}B^2_{ij}(t)}{\sqrt{2}}\right) & \text{si} \quad i > j \end{cases}$$

où $(B_{ii})_{1\leq i\leq m}, (B^1_{ij})_{1\leq j<i\leq m}, (B^2_{ij})_{1\leq j<i\leq m}$ sont trois familles indépendantes de mouvements browniens tels que :

$$< dB_{ii}, dB_{kk} >_t = \left(1 - \frac{\beta}{2}\right)dt := (1-\rho)dt \qquad 1 \leq i \neq k \leq m,$$

alors que l'indépendance est requise pour les deux autres familles. Chacune des valeurs propres de ce processus satisfait l'EDS ci-dessus sauf que les ν_i, $1 \leq i \leq m$ sont corrélés de la même façon que les B_{ii}, $1 \leq i \leq m$. Ce processus coincide avec le mouvement brownien de Dyson pour $\beta = 2$. D'une manière générale, il peut s'exprimer en fonction de ce dernier, disons K, de la façon suivante :

$$X_t = \sqrt{\rho}K_t + \theta_0 \,\mathrm{tr}(K_t)I_m$$

où θ_0 est une racine de $m\theta^2 + 2\sqrt{\rho}\theta - (1-\rho) = 0$. Cependant, la corrélation fait que le temps de collision $\tau := \inf\{t,\ \lambda_i(t) = \lambda_j(t) \text{ pour un couple } (i,j)\}$ entre deux valeurs propres de X est infini presque sûrement, ce qui n'est pas le cas des valeurs propres de K qui peuvent se toucher si $0 < \beta < 1$.

3.4. Résumé du chapitre 6 : Free Jacobi process. Ce travail rentre dans le cadre des probabilités libres et plus exactement des processus libres définis comme limite, dans un sens bien déterminé, de processus matriciels. Cette limite est au sens des moments non commutatifs : si $X(m)$ désigne le processus matriciel de taille m, le processus limite X est défini par

$$\lim_{m\to\infty}\mathbb{E}(\mathrm{tr}_m(X_{t_1}(m)\ldots X_{t_k}(m)) = \Phi(X_{t_1}\ldots X_{t_1})$$

pour toute collection $t_1, \ldots, t_k$, où tr est la trace normalisée et Φ est une forme linéaire sur une algèbre $\mathscr{A}$ qui sera l'espace d'état de X. Il est connu dans la littérature que le mouvement brownien hermitien converge vers **le mouvement brownien libre additif** ([**111**]), le mouvement brownien unitaire vers **le mouvement brownien libre multiplicatif** ([**12**]) et le processus de Laguerre vers **le processus de Wishart libre** ([**24**]). Le but est d'établir une EDS libre pour la limite du processus de Jacobi complexe : le processus de Jacobi libre.
Rappelons tout d'abord la construction du processus de Jacobi complexe $J(m)$

([**43**]) : soit $Y(d)$ une matrice brownienne unitaire de taille d. Celle ci peut être définie via l'équation de la châleur ou bien comme un processus unitaire partant de la matrice identité et ayant des acroissements (à gauche ou à droite) indépendants et stationnaires ([**12**]). Soit $X_{m,p}$ le coin supérieur gauche $m \times p$ de $Y(d)$. Alors $J(m) := X_{m,p}X^{\star}_{m,p}$ et on peut écrire

$$J(m) \oplus 0_{n-m} = (X_{m,p}X^{\star}_{m,p})(m) \oplus 0_{n-m} = P_mY(d)Q_pY^{\star}(d)P_m$$

où P_m et Q_p sont les projections

$$P_m = \begin{pmatrix} I_m & \\ & 0 \end{pmatrix}, \qquad Q_p = \begin{pmatrix} I_p & \\ & 0 \end{pmatrix}$$

Ce processus est l'unique solution forte de l'EDS suivante :

$$dJ_t = \sqrt{J_t}dB_t\sqrt{I_m - J_t} + \sqrt{I_m - J_t}dB^{\star}_t\sqrt{J_t} + 2(pI_m - (p+q)J_t)dt$$

où B est une matrice brownienne complexe $m \times m$ et $q = d - p$ tels que $p \wedge q \geq m$. Si $d = d(m)$ et $p = p(m)$ sont tels que :

$$\lambda := \lim_{m\to\infty} \frac{m}{p(m)} > 0, \qquad \theta := \lim_{m\to\infty} \frac{m}{d(m)} \in]0,1],$$

alors on montre dans un premier temps que $J(m)$ converge au sens indiqué ci-dessus vers un processus libre qu'on appellera *processus de Jacobi libre* partant de $J_0 = PQP$ et on notera J. J s'écrit sous la forme $PYQY^{\star}P$ où Y est le mouvement brownien multiplicatif et P et Q sont respectivement deux projecteurs tels que $\Phi(P) = \lambda\theta$ et $\Phi(Q) = \theta$. Cette écriture permet de voir que J peut être considéré comme un processus dans *l'espace compressé* $P\mathscr{A}P$ muni de la forme $\tilde{\Phi} := (1/\Phi(P))\Phi$. Le processus $\tilde{J} := PYZQZ^{\star}Y^{\star}P$ où Z est une variable unitaire libre avec $\{Y, Y^{\star}, P, Q\}$, définit le processus de Jacobi libre partant de $\tilde{J}_0 = PZQZ^{\star}P$. On notera J au lieu de $\tilde{J}$ pour alléger les notations. Dans un second temps, on utilise la formule d'Itô libre [**13**]) ainsi que l'EDS libre satisfaîte par Y pour établir sous réserve d'injectivité de J et de $P - J$ dans $P\mathscr{A}P$ que :

$$dJ_t = \sqrt{\lambda\theta}\sqrt{P - J_t}dW_t\sqrt{J_t} + \sqrt{\lambda\theta}\sqrt{J_t}dW^{\star}_t\sqrt{P - J_t} + (\theta P - J_t)\,dt$$

où W est un mouvement brownien complexe libre ([**24**]). Il est à noter que si J est un processus de Jacobi de paramètres λ, θ alors $P - J$ l'est aussi mais avec les paramètres $\lambda\theta/(1-\theta)$, $1 - \theta$.

Le reste du papier est consacré à la détermination des valeurs de λ, θ pour lesquelles la condition d'injectivité requise est assurée. Deux cas sont considérés :

– **Le cas stationnaire** : Y suit la loi donnée par la mesure de Haar et la loi de J_t ne dépend plus de t. De plus, la matrice $J_t(m)$ n'est autre qu'une matrice de Jacobi de loi Beta multivariée ([**45**]) et la loi du processus limite J figure déjà dans [**23**]. Néanmoins, on utilise une technique différente pour retrouver sa transformée de Cauchy et on détaille le calcul de la mesure spectrale.

L'injectivité est assurée pour $\lambda \in]0,1]$ et $1/\theta \geq \lambda + 1$. On a même que le processus est inversible pour des inégalités strictes.

- **Le cas général** : On arrive à étendre le résultat d'injectivité mais la situation est loin d'être facile. Ceci requiert en plus l'inversibilité de J_0 et $P - J_0$ dans $P\mathscr{A}P$. Pour cela, on s'est inspiré du cas matriciel dans lequel on a souvent recours à la fonctionnelle $\log\det$. Plus précisémment, on considère $\tilde{\Phi}(\log(P - J_t))$ qui est définie par le calcul fonctionnel. Pour aborder les calculs, on établit une relation de récurrence pour les moments $\tilde{\Phi}(J_t^n)$, $n \geq 2$. On montre alors que pour ces valeurs de λ, θ, $\tilde{\Phi}(\log(P - J_t)) + (1 - \lambda\theta)t > \tilde{\Phi}(\log(P - J_0))$. On utilise ensuite le fait que $P - J$ est encore un processus de Jacobi.
  On exploite une seconde fois la formule de récurrence : on montre dans le cas stationnaire que pour $\lambda = 1, \theta = 1/2$ (la loi de J_t est alors une loi Beta $(1/2, 1/2)$) et si T_k désigne le polynôme de Chebycheff de première espèce de degré k défini par $T_k(x) := \cos(k \arccos x)$ ([**3**]), alors $T_k(2P - J)$ est une martingale libre pour la filtration naturelle du processus. Enfin, une EDP est établie pour la transformée de Cauchy de la loi de J_t qui est en accord avec le cas stationnaire.

3.5. Résumé du chapitre 7. On continue avec le processus de Jacobi mais cette fois on se contente du cas univarié. Pour des raisons de cohérence, nous introduisons des notations différentes de celles utilisées avant. Ce processus est l'unique solution forte de

$$dJ_t = 2\sqrt{J_t(1 - J_t)}dW_t + (d - (d + d')J_t)dt, \quad d, d' \geq 0.$$

Comme nous l'avons signalé, le lecteur peut rencontrer deux autres définitions équivalentes à la précédente : elles sont obtenues par un changement de l'espace d'état et (ou) un changement de temps déterministe. La première a été utilisée par M. Zani dans sa thèse ([**117**]) :

$$dY_t = \sqrt{1 - Y_t^2}\, dW_t + (bY_t + c)dt.$$

avec $d = 2(c - b)$, $d' = -2(c + b)$. La deuxième est celle qui figure dans [**114**] et qui n'est autre que celle du processus $(X_t := Y_{2t})_{t \geq 0}$ donnée par son générateur :

$$\mathscr{L} = (1 - x^2)\frac{\partial^2}{\partial^2 x} + (px + q)\frac{\partial}{\partial x}, \qquad x \in [-1, 1]$$

avec $d = q - p$, $d' = -(p + q)$. Notre première intention était de donner une expression plus simple de la densité du semi groupe de $(X_t)_{t \geq 0}$ qui est sous la forme

$$p_t(x, y) = \sum_{n \geq 0} e^{-r_n t} P_n^{\alpha,\beta}(x) P_n^{\alpha,\beta}(y) W(y), \quad x, y \in [-1, 1],$$

où $P_n^{\alpha,\beta}$ désigne le polynôme de Jacobi normalisé de degré n et de paramètres $\alpha, \beta > -1$ ([**3**]) et

$$\begin{aligned} r_n &= n(n+\alpha+\beta+1) \\ W(y) &= \frac{(1-y)^\alpha(1+y)^\beta}{2^{\alpha+\beta+1}B(\alpha,\beta)}, \end{aligned}$$

B étant la fonction Beta d'Euler. De plus, $d = 2(\beta+1)$, $d' = 2(\alpha+1)$. Notre technique est basée sur la subordination de X par un temps aléatoire $(T_t)_{t\geq 0}$ convenablement choisi. Ceci se ramène à calculer la transformée de Laplace en r_n de la densité ν_t de T_t. Ainsi, la nouvelle densité, disons $q_t(x,y)$, s'écrit :

$$q_t(x,y) = \sum_{n\geq 0} e^{-nt} P_n^{\alpha,\beta}(x) P_n^{\alpha,\beta}(y) W(y) := \sum_{n\geq 0} r^n P_n^{\alpha,\beta}(x) P_n^{\alpha,\beta}(y) W(y), \quad r = e^{-t}$$

La mesure ν_t dépend de deux paramètres $\mu > 0$, $\delta > 0$ et a pour transformée de Laplace ([**1**]) :

$$\int_0^\infty e^{-us} \nu_t^{\mu,\delta}(ds) = e^{-t\delta(\sqrt{2u+\mu^2}-\mu)}$$

de laquelle on récupère la densité

$$f_t(s) = \frac{\delta t}{\sqrt{2\pi}} e^{\delta t \mu} s^{-3/2} \exp\left(-\frac{1}{2}(\frac{t^2\delta^2}{s} + \mu^2 s)\right) \mathbf{1}_{\{s>0\}}$$

Cette densité correspond au subordinateur $(T_t^{\mu,\delta})_{t\geq 0}$ défini par

$$T_t^{\mu,\delta} := \inf\{s, B_s + \mu s = \delta t\}$$

où B est un mouvement Brownian standard. Il est facile de voir qu'il faut prendre $\delta = 1/\sqrt{2}$, $\mu = (\alpha+\beta+1)/\sqrt{2}$ pour $\alpha+\beta > -1$. D'une part, la densité $q_t(x,y)$ fait intervenir le produit des fonctions

$$t \mapsto (1/\cosh(t/2))^h,\ h = h(\alpha,\beta) > 0, \qquad t \mapsto \tanh(t/2)/(t/2).$$

D'autre part, il faut observer, à partir de l'expression de f_t que $t \mapsto e^{-(\alpha+\beta+1)t/2} q_t$ est la transformée de Laplace en $t^2/4$ (à une constante prés) de

$$s \mapsto p_{2/s}(x,y) s^{-1/2} e^{-\mu^2/2s}.$$

Il nous reste alors à inverser les transformées de Laplace des fonctions hyperboliques ci-dessus, tâche qui a été achevée dans [**16**] et [**96**]. A partir de là, on arrive à une expression faisant intervenir un seul $P_n^{\alpha,\beta}$ mais avec une dépendance en temps un peu compliquée. Néanmoins, et dans le cas ultrasphérique $\alpha = \beta > -1/2$, les facteurs sont plus simples, et spécialement pour $x = 0$. De plus, $\alpha = \beta \Rightarrow d = d' \Rightarrow c = 0$, $d = -2b$. Ceci nous a permis de résoudre un problème de grandes déviations pour une famille d'estimateurs $\{\hat{b}_t\}_{t\geq 0}$ de b, chacun est basé sur l'observation d'une trajectoire de Y jusqu' à l'instant t ([**117**]). Nous n'allons pas donner les détails du calcul mais nous mentionnons quand même que le théorème de M.

Zani rentre dans un cadre de grandes déviations qui n'est pas classique : en effet, la transformée log-Laplace qu'on calcule peut être non-escarpée.

CHAPITRE 2

Useful Definitions and Notations

We present some facts we make use of later. The first part concerns multivariate special functions while the second one collects algebraic tools needed in free probability.

1. Multivariate Special Functions

1.1. Jack polynomials. Let τ be a partition of **weight** k and **length** m, that is $\tau = (k_1 \geq \cdots \geq k_m)$ and $|\tau| := k_1 + \cdots + k_m = k$. Let

$$\rho_\tau = \sum_{i=1}^{m} k_i(k_i - 1 - \beta(i-1))$$

The Jack polynomial ([**86**]) $J_\tau^{(2/\beta)}(x_1, \ldots, x_m)$, $\beta > 0$ is defined as the unique (up to normalization) homogenous (of degree k) symmetric eigenfunction of the operator :

$$\sum_{i=1}^{m} x_i^2 \partial_i^2 + \beta \sum_{i \neq j} \frac{x_i^2}{x_i - x_j} \partial_i.$$

corresponding to the eigenvalue $\rho_\tau + k(m-1)$. The Dyson index β is referred to *the inverse Jack parameter*. Several normalizations are used in the literature and the one adopted here and in papers we refer to is specified by requiring :

$$(x_1 + \cdots + x_m)^k = \sum_{|\tau|=k} J_\tau^{(2/\beta)}(x_1, \ldots, x_m).$$

For $\beta = 1$, this is the so-called *zonal polynomial* ([**89**]). For $\beta = 2$, this is (up to a normalization) *the Schur function* defined by ([**62**], [**86**]) :

$$s_\tau(x_1, \ldots, x_m) = \frac{\det(x_i^{k_j+m-j})}{\det(x_i^{m-j})}$$

For both cases, $x_1, \cdots, x_m$ can be viewed as eigenvalues of real symmetric or complex Hermitian matrices respectively. In this way, a wide literature is developed using the Haar measure on orthogonal and unitary groups acting on the corresponding matrix spaces ([**89**], [**62**]). A different approach was investigated by Faraut and Korànyi who dealt with spherical functions on Jordan algebras and recovered well known results on this topic by specializing to particular algebras ([**52**]).

1.2. Multivariate Gamma function and generalized Pochammer symbol. The univariate Gamma function is defined as a one parameter-dependent integral on the positive half-line :

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt,\, x > 0.$$

It extends to a meromorphic function on $\mathbb{C}\setminus\mathbb{Z}_-$. The Pochammer symbol is defined by

$$(a)_k = (a+k-1)(a+k-2)\dots(a+1)a.$$

When it makes sense, this writes $\Gamma(a+k)/\Gamma(a)$. For negative integer, $(-n)_k = 0$ for some k.

Notations :

- H_m : $m \times m$ Hermitian matrices space.
- H_m^+ : $m \times m$ positive Hermitian matrices space.
- $\tilde{H}_m^+$: $m \times m$ positive definite Hermitian matrices space.

On H_m, the *multivariate Gamma function* is defined by ([**69**]) :

$$\Gamma_m(a) = \int_{\tilde{H}_m^+} e^{-\operatorname{tr}(z)}det(z)^{a-m}dz, \quad a > m-1,\, z = x + \sqrt{-1}y,$$

where $dz = \prod_{i\leq j} dx_{ij} \prod_{i<j} dy_{ij}$ is the **Lebesgue measure** on H_m. Analogous definition is given on the space of symmetric matrices ([**89**]). A more general one is considered in [**52**] where the integration range is a *symmetric cone*. In the Hermitian case, one has :

$$\Gamma_m(a) = \pi^{m(m-1)/2}\prod_{i=1}^{m}\Gamma(a-i+1)$$

and a similar formula holds in the real symmetric case :

$$\Gamma_m(a) = \pi^{m(m-1)/4}\prod_{i=1}^{m}\Gamma\left(a - \frac{1}{2}(i-1)\right)$$

With regard to both formulas, one can define the multivariate Gamma function associated to the Jack parameter β by ([**70**]) :

$$\Gamma_m^{(\beta)}(a) = \pi^{\beta m(m-1)/4}\prod_{i=1}^{m}\Gamma\left(a - \frac{\beta}{2}(i-1)\right), \quad \beta > 0.$$

Given a partition τ of length m, the *generalized Pochammer symbol* is defined by ([**5**]) :

$$(a)_\tau^{(\beta)} = \prod_{i=1}^{m}\left(a - \frac{\beta}{2}(i-1)\right)_{k_j}.$$

For sake of clarity, we will omit β when writing this symbol as well as $\Gamma_m^{(\beta)}$.

1.3. Multivariate Hypergeometric functions. Let $p, q \in \mathbb{N}$ and $x = (x_1, \dots, x_m)$. These are defined by ([**70**]) :

$$ {}_pF_q^{(2/\beta)}((a_i)_{1\leq i\leq p}, (b_j)_{1\leq j\leq q}; x) = \sum_{k=0}^{\infty}\sum_{|\tau|=k} \frac{(a_1)_\tau \cdots (a_p)_\tau}{(b_1)_\tau \cdots (b_q)_\tau} \frac{J_\tau^{(2/\beta)}(x)}{k!} $$

provided that $b_j - (\beta/2)(i-1)$, $1 \leq j \leq q$, $1 \leq i \leq m$, is neither negative nor zero. For $m = 1$, this reduces to the univariate hypergeometric series (see [**85**]) :

$$ {}_p\mathscr{F}_q((a_i)_{1\leq i\leq p}, (b_j)_{1\leq j\leq q}; x) = \sum_{k=0}^{\infty} \frac{(a_1)_\tau \cdots (a_p)_\tau}{(b_1)_\tau \cdots (b_q)_\tau} \frac{x^k}{k!} $$

When $p = q + 1$, both series converge for $||x|| < 1$ and diverge for $||x|| > 1$. When $p < q$, they converge for all $x \in \mathbb{R}^m$. Else, they diverge ([**8**], [**70**]) unless it terminates. For $p = 2$, $q = 1$, it is the so-called *Gauss hypergeometric function*. It is the unique symmetric eigenfunction of

$$ \sum_{i=1}^{m} x_i(1-x_i)\partial_i^{2,x} + \beta\sum_{i\neq j}\frac{x_i(1-x_i)}{x_i - x_j}\partial_i^x + \sum_{i=1}^{m}\left[b_1 - \frac{\beta}{2}(m-1) - \left(a_1 + a_2 + 1 - \frac{\beta}{2}(m-1)\right)x_i\right]\partial_i^x $$

associated to the eigenvalue ma_1b_1 and that equals to 1 at 0 (see [**8**] p. 585 or [**70**] p. 1097). For $p = q = 1$, it is the *confluent hypergeometric function* which can be recovered from ${}_2F_1^{(2/\beta)}$ in the following way

$$ {}_1F_1^{(2/\beta)}(a_1, b_1; x) = \lim_{a_2\to\infty} {}_2F_1^{(2/\beta)}(a_1, a_2, b_1; \frac{x}{b_1}) $$

With regard to the product defining the generalized Pochammer symbol, one can guess that the hypergeometric function simplifies to a polynomial for specific values of $(a_i)_{1\leq i\leq p}$ depending on β. In cases $\beta = 1, 2$, Jack polynomials reduce to zonal polynomials and Schur functions respectively. As mentioned before, the latters correspond to some underlying matrice ensembles. That is why functions above are called in these cases *of matrix argument*. The reader should notice that owing to orthogonal and unitary groups, ${}_pF_q^{(2)}$ and ${}_pF_q^{(1)}$ are more handable than those corresponding to other Jack parameters ([**89**], [**62**]). Even more, in the complex case, one has determinantal representations involving univariate functions ([**62**]). Similar results hold for multivariate orthogonal polynomials as well ([**7**], [**81**], [**82**], [**83**]).
The hypergeometric function *of two arguments* is defined by :

$$ {}_pF_q^{(2/\beta)}((a_i)_{1\leq i\leq p}, (b_j)_{1\leq j\leq q}; x,\, y) = \sum_{k=0}^{\infty}\sum_{|\tau|=k} \frac{(a_1)_\tau \cdots (a_p)_\tau}{(b_1)_\tau \cdots (b_q)_\tau} \frac{J_\tau^{(2/\beta)}(x)J_\tau^{(2/\beta)}(y)}{J_\tau^{(2/\beta)}(1_m)k!} $$

where $1_m = (1, \dots, 1)$. When $\beta = 1, 2$, x, y can be viewed as eigenvalues vectors of symmetric and Hermitian matrices. In this context, they are known as a function *of two matrix arguments*. In the latter case, a determinantal representation holds

([**37**]). Recently, Professor D. Richards tolds us that this was proved in [**63**] and that authors discovered however that it is due to Khatri ([**76**]).

Remark. In [**5**], authors use α and $C_\tau^{(\alpha)}$ to denote the Jack parameter and Jack polynomials respectively. Keeping this in mind, one notices that $\beta = 2/\alpha$.

1.4. Modified Bessel function of index ν. ([**85**])

$$I_\nu(x) := \sum_{k=0}^{\infty} \frac{1}{\Gamma(\nu+k+1)k!}\left(\frac{x}{2}\right)^{2k+\nu} = \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu} {}_0\mathscr{F}_1\left(\nu+1; \frac{x^2}{4}\right)$$

2. $C^\star$ and von Neumann Algebras

We refer the reader to [**38**] and [**39**] for facts on algebras. A non-commutative probability space is a pair $(\mathscr{A}, \Phi)$ where $\mathscr{A}$ is a *unital algebra* and

$$\Phi: \quad \mathscr{A} \mapsto \mathbb{C}, \quad \Phi(\mathbf{1}) = 1$$

a linear functional called *state*.

Examples

(1)
$$\mathscr{A}_m = \bigcap_{p>0} L^p(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}) \times \mathbb{M}_m(\mathbb{C})$$
the set of $m \times m$ random matrices with all order finite moments endowed with the normalized trace expectation :
$$\Phi_m := \frac{1}{m}\mathbb{E}(tr) := \mathbb{E}(tr_m)$$

(2) $B(H)$: the set of bounded linear operators on a Hilbert space H with the *pure state* $\Phi(a) =< ax, x >$, $a \in B(H)$, where $x \in H$ with unit norm.

- An *involutive Banach algebra* is equipped with an *involution* $\star$ and a norm $||\cdot||$ s.t $||a^\star|| = ||a||$, $a \in \mathscr{A}$, the algebra is complete with respect to $||\cdot||$.
- A $C^\star$*-algebra* is an involutive Banach algebra s. t. $||aa^\star|| \geq ||a||^2$ for all $a \in \mathscr{A}$. This implies that $||aa^\star|| = ||a||^2$. Note that in a $C^\star$-algebra, all states are of the form given in the second example (GNS representation) : there exist a Hilbert space H, a representation π of $\mathscr{A}$ in H and a unit norm element $x \in H$ such that $\Phi(a) =< \pi(a)x, x >$, $a \in \mathscr{A}$.
- A *von Neumann algebra* is a subalgebra $B(H)$ (the algebra of bounded operators acting on a Hilbert space H) which is closed with respect to the weak topology. The von Neumann commutant Theorem asserts that the bicommutant $A^{''} = A$ (recall that the commutant $A^{'}$ of A is the set of elements that commute with the elements of A). Moreover, the tensor product of von Neumann algebras is still a von Neumann algebra.

The state Φ can be :
- *tracial* : $\quad \Phi(ab) = \Phi(ba)$

– *faithful* : $\Phi(aa^\star) = 0 \Rightarrow a = 0.$

– *normal* : $\Phi(\sup_i a_i) = \sup_i \Phi(a_i),$

for all filtered bounded family $(a_i)_i \in \mathscr{A}$. In example (1), involution has to be the usual adjonction and properties above are obviously fulfilled. However, in example (2), one needs some restrictions (see [**12**] for details). As in classical probability, we endow our space with a family $(\mathscr{A}_t)_{t\geq 0}$ of increasing subalgebras called filtration. When $\mathscr{A}$ is a $C^\star$ or a von Neumann algebra and Φ is tracial, there exists a unique conditional expectation that we shall denote by $\Phi(\cdot/\mathscr{A}_s)$, $s \leq t$.

CHAPITRE 3

Laguerre Process and generalized Hartman-Watson Law

This chapter is a detailed version of the paper that will appear in Bernoulli Journal Volume 13, no. 2, p. 556-580.

1. Introduction

Real and complex Wishart matrices have been extensively studied along the years by many statisticians such as Chikuze, James, Letac, Massam, Muirhead and others ([**89**], [**69**], [**31**]). They trace back to Wishart ([**113**]) who used them in multivariate statistical analysis as sample covariance matrices (see also the introduction). The number of columns counts the variates and the number of rows is the sample size. Furthermore, in Bayesian statistic, the Wishart law is known to be a conjuguate family, that is prior and posterior distributions belong to the same family. Then, a dynamic counterpart, called "time-dependent Wishart matrices", appeared in physical literature ([**30**]) imitating the Hermitian Brownian motion of Dyson. In the early ninetees, a probabilistic setting of these matrices was taken at hand by Bru ([**19**]) replacing the multivariate $n \times m$ normal distribution by a $n \times m$ matrix Brownian motion, say $(B_t)_{t\geq 0}$. The process $(X_t)_{t\geq 0}$ is defined by $X_t = B_t^T B_t$ and denoted by $W(n, m, X_0)$. m is the size of $(X_t)_{t\geq 0}$, n is its dimension and X_0 its the starting point. For $n \geq m$, it satisfies the stochastic differential equation (SDE) below :

$$\begin{aligned} dX_t &= B_t^T dB_t + dB_t^T B_t + nI_m dt \\ &= \sqrt{X_t} dN_t + dN_t^T \sqrt{X_t} + nI_m dt, \qquad X_0 = B_0^T B_0 \end{aligned}$$

where I_m denotes the unit matrix, the superscript ^{T} stands for the transpose, $\sqrt{X_t}$ is the matrix square root of the positive definite matrix X_t and $(N_t)_{t\geq 0}$ is a $m \times m$ Brownian matrix. Following the one dimensional case, this suggests to define the $W(\delta, m, X_0)$ as the unique solution of the latter SDE with δ instead of n. Unfortunately, this was shown to hold for δ in the Gindikin ensemble defined by $\{1, \ldots, m-1\} \cup]m-1, \infty[$. Thus, it can be viewed as an extension of the squared Bessel process to higher dimension. In this way, Donati et al. ([**40**]) tried to derive multivariate analogs of well known properties : absolute-continuity relations, generalized Hartman-Watson law defined by mean of its Laplace transform (see [**115**] for the univariate case) , the first hitting time of 0 as well as its tail distribution when finite. Expressions obtained there involve multivariate special functions of real symmetric argument, such as Gamma and hypergeometric functions (see [**89**] for definitions). However, the latters, being defined in terms of *zonal polynomials*, are quite complicated to deal with and to our best knowledge, there are no more precise results on the law of these variables. Nevertheless, in the complex case, things seem easier than they were in the real case and this was at the origin of this work. Indeed, hypergeometric functions of complex Hermitian argument can be expressed as a determinant of a matrix whose entries are univariate hypergeometric functions. More precisely, the following is due to Gross and Richards ([**62**]) :

$$ {}_pF_q^{(1)}((a_1,\dots,a_p,b_1,\dots,b_q;X) = \frac{det(x_i^{m-j}\,{}_p\mathscr{F}_q(a_1-j+1,\dots,a_p-j+1,\dots,b_q-j+1;x_i))}{V(X)} $$

where X is a $m \times m$ complex Hermitian matrix with eigenvalues (x_i) and V is the Vandermonde function. This results from the fact that the corresponding Jack polynomial of Jack parameter $\beta = 2$ fits the (normalized) *Schur functions* and one of the famous Hua's formulas ([**51**] p. 198). This together with some properties of the univariate functions will allow us to deepen our results at least when $m = 2$. The rest of this paper consists of seven sections, which are respectively devoted to the following topics : in section 2, we introduce the Laguerre process of integer dimension and compute the infinitesimal generator. Section 3 is concerned with the behaviour of the eigenvalue process from which we deduce the strict positivity of X_t when $n \geq m$ and that eigenvalues never collide. Then, in section 4, existence and uniqueness results of Laguerre processes of positive real dimensions are proved and previous results extend to this setting. At this end, we follow [**19**] and [**20**] with minor modifications. Section 5 treats absolute-continuity relations, the Laplace transform of the so-called *generalized Hartman-Watson* law as well as the tail distribution of T_0, the first hitting time of 0. Finally, we investigate the particular case $m = 2$: we invert this Laplace transform and compute the density of $S_0 := 1/(2T_0)$.

2. Laguerre Process of Integer Dimension

Let B be a $n \times m$ complex Brownian matrix starting from B_0 , ie, $B = (B_{ij})$ where the entries B_{ij} are independent complex Brownian motions, so we can write $B = B^1 + iB^2$. We are interested in the matrix-valued process $X_t := B_t^\star B_t$. Itô's formula leads to :

$$ dX_t = dB_t^\star B_t + B_t^\star dB_t + 2nI dt \tag{14} $$

Definition. $(X_t)_{t\geq 0}$ is called the Laguerre process of size m, of dimension n and starting at $X_0 = B_0^\star B_0$, and will be denoted by $L(n, m, X_0)$.

Remarks. 1/ For $m = 1$, $(X_t)_{t\geq 0}$ is a squared Bessel process of dimension $BESQ(2n, X_0)$.
2/ Set $X = (x_{ij})_{i,j}$. We can easily check that

$$ d(x_{ii}(t)) = 2\sqrt{x_{ii}(t)}d\gamma_i(t) + 2ndt \qquad 1 \leq i \leq m, $$

where (γ_i) are independent Brownian motions, thus, each diagonal term is a $BESQ(2n, X_{ii}(0))$. Summing over i gives :

$$d(\mathrm{tr}(X_t)) = 2\sqrt{\mathrm{tr}(X_t)}d\beta_t + 2nmdt \tag{15}$$

where β is a Brownian motion. Consequently, $(\mathrm{tr}(X_t))$ is a $BESQ(2nm, \mathrm{tr}(X_0))$ of dimension $2n$ starting from $\mathrm{tr}(X_0)$.

4/ We can deduce from equation (1) that for every $i, j, k, l \in \{1, \cdots, m\}$:

$$< dx_{ij}, dx_{kl} >_t = 2(x_{il}\delta_{kj} + x_{kj}\delta_{il})dt$$

Note that this is different from (I-1-5) in [**19**] : the difference is due to the fact that, for a complex Brownian motion γ, one has $d\langle\gamma, \gamma\rangle_t = 0$ and $d\langle\gamma, \overline{\gamma}\rangle_t = 2t$.

2.1. Infinitesimal generator. On the space of complex Hermitian argument functions, we define the first order matrix-valued differential operators :

$$\frac{\partial}{\partial x} := \left(\frac{\partial}{\partial x_{jk}}\right)_{j,k}, \qquad \frac{\partial}{\partial y} := \left(\frac{\partial}{\partial y_{jk}}\right)_{j,k}, \qquad \frac{\partial}{\partial z} := \left(\frac{\partial}{\partial x_{jk}} - i\frac{\partial}{\partial y_{jk}}\right)_{j,k},$$

Second order matrix-valued operators are define via matrix multiplication rule :

$$\left(\frac{\partial}{\partial z}\right)^2_{ij} := \sum_k \frac{\partial^2}{\partial z_{ik}\partial z_{kj}}, \quad \left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right)_{ij} := \sum_k \frac{\partial^2}{\partial x_{ik}\partial y_{kj}}$$

Proposition 2.1. *Let functions f satisfying :*

$$\frac{\partial f}{\partial x_{ij}} = \frac{\partial f}{\partial x_{ji}}, \quad \frac{\partial f}{\partial y_{ij}} = -\frac{\partial f}{\partial y_{ji}} \qquad \textit{for all} \quad i, j.$$

Then, for such f, the infinitesimal generator of a Laguerre process $L(n, m, x)$ is given by :

$$\mathscr{L} = 2n\ \mathrm{tr}(\Re\left(\frac{\partial}{\partial z}\right)) + 2[\mathrm{tr}(x\Re\left(\frac{\partial}{\partial z}\right)^2) + \mathrm{tr}(y\Im\left(\frac{\partial}{\partial z}\right)^2)] \tag{16}$$

where $\partial/\partial z$ is the operator defined above.

Remark. Using the fact that $x^T = x$, $y^T = -y$ and $\mathrm{tr}(AB) = \mathrm{tr}(BA) = \mathrm{tr}(B^T A^T)$ for any two matrices A and B, it follows that :

$$\mathrm{tr}\left(y\frac{\partial}{\partial y}\frac{\partial}{\partial x}\right) = \mathrm{tr}\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}y\right) = \mathrm{tr}\left(y\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right),$$

thus,

$$\mathrm{tr}\left(y\Im(\frac{\partial}{\partial z})^2\right) = 2\,\mathrm{tr}\left(y\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right)$$

Proof : Let us first note that :

$$\left(\frac{\partial}{\partial z}\right)^2 = \left(\frac{\partial}{\partial x}\right)^2 - \left(\frac{\partial}{\partial y}\right)^2 - i\left(\frac{\partial}{\partial y}\frac{\partial}{\partial x} + \frac{\partial}{\partial x}\frac{\partial}{\partial y}\right),$$

Now, consider two functions f and g defined on H_m and on $\mathbb{M}(n,m)$ respectively, then :

$$g(B_t) = f(B_t^\star B_t) \quad \Rightarrow \quad \frac{1}{2}\Delta g = \mathscr{L} f$$

where Δ denotes the Laplacian operator and $\mathscr{L}$ writes

$$\begin{aligned}
\mathscr{L} &= 2n\sum_{k=1}^{m}\frac{\partial}{\partial x_{kk}} + 2\sum_{p,j=1}^{m} x_{jp}\sum_{i=1}^{m}\frac{\partial^2}{\partial x_{ij}\partial x_{ip}} + 2\sum_{p,j=1}^{m} x_{jp}\sum_{i=1}^{m}\frac{\partial^2}{\partial y_{ij}\partial y_{ip}} \\
&+ 2\sum_{p,j=1}^{m} y_{jp}\sum_{i=1}^{m}\frac{\partial^2}{\partial x_{ij}\partial y_{ip}} + 2\sum_{p,j=1}^{m} y_{jp}\sum_{i=1}^{m}\frac{\partial^2}{\partial y_{ip}\partial x_{ij}} \\
&= 2n\sum_{k=1}^{m}\frac{\partial}{\partial x_{kk}} + 2\sum_{p,j=1}^{m} x_{jp}\left((\frac{\partial}{\partial x})^T\frac{\partial}{\partial x}\right)_{pj} + 2\sum_{p,j=1}^{m} x_{jp}\left((\frac{\partial}{\partial y})^T\frac{\partial}{\partial y}\right)_{pj} \\
&+ 2\sum_{p,j=1}^{m} y_{jp}\left((\frac{\partial}{\partial y})^T\frac{\partial}{\partial x}\right)_{pj} - 2\sum_{p,j=1}^{m} y_{pj}\left((\frac{\partial}{\partial x})^T\frac{\partial}{\partial y}\right)_{jp}
\end{aligned}$$

Conditions $\dfrac{\partial f}{\partial x_{jp}} = \dfrac{\partial f}{\partial x_{pj}}$ and $\dfrac{\partial f}{\partial y_{jp}} = -\dfrac{\partial f}{\partial y_{pj}} \Rightarrow \left(\dfrac{\partial}{\partial x}\right)^T = \dfrac{\partial}{\partial x}$ and $\left(\dfrac{\partial}{\partial y}\right)^T = -\dfrac{\partial}{\partial y}$ and (3) follows.

3. Eigenvalues Process

In the sequel, we will suppose that $n \geq m$. The following result was derived in [**78**] and [**74**] with no proof (see also [**19**] for the real case) :

THEOREM 3.1. *Let $\lambda_1(t), \cdots, \lambda_m(t)$ denote the eigenvalues of X_t. Suppose that at time $t = 0$, all the eigenvalues are distinct. Then, the eigenvalues process $(\lambda_1, \ldots, \lambda_m)$ satisfies the following stochastic differential system :*

$$d\lambda_i(t) = 2\sqrt{\lambda_i(t)}\, d\beta_i(t) + 2\left[n + \sum_{k\neq i}\frac{\lambda_i(t)+\lambda_k(t)}{\lambda_i(t)-\lambda_k(t)}\right] dt \qquad 1 \leq i \leq m, \qquad t < \tau,$$

where the $(\beta_i)_{1\leq i\leq m}$ are independent Brownian motions and τ is the first collision time defined by $\tau := \inf\{t, \lambda_i(t) = \lambda_j(t)\, for\ some\, (i,j)\}$.

Proof : The proof is similar to the one given in (cf [**19**]) with slight modifications. Before proceeding, some notations are needed.

Notations. Given three matrix-valued semimartingales A, B, and C, we set :

$$\begin{aligned}
(dA_t)(dB_t) &= \left(\sum_k < da_{ik}(t), db_{kj}(t) >\right)_{i,j} \\
(dA_t)C_t(dB_t) &= \left(\sum_{k,l} c_{kl} < da_{ik}(t), db_{lj}(t) >\right)_{i,j},
\end{aligned}$$

where a, b and c are respectively the entries of A, B and C. Thus, the Itô's formula writes :

$$A_tB_t = A_tdB_t + dA_tB_t + (dA_t)(dB_t).$$

Now, we go back to the proof. Since X is Hermitian, there exists a continuous unitary process H which diagonalizes X, that is :

$$H_t^\star X_t\, H_t = D_t := \mathrm{diag}(\lambda_i(t)). \tag{17}$$

Note that the H_t's entries are continuous semimartingales since they are C^∞-functions in (x_{ij}). The Itô's formula gives :

$$\begin{aligned}
dD_t &= dH_t^\star X_tH_t + H_t^\star X_tdH_t + H_t^\star dX_tH_t + (dH_t^\star)(dX_t)H_t + H_t^\star(dX_t)(dH_t) + (dH_t^\star)X_t(dH_t) \\
&= dH_t^\star H_tD_t + D_tH_t^\star dH_t + H_t^\star dX_tH_t + (dH_t^\star)(dX_t)H_t + H_t^\star(dX_t)(dH_t) + (dH_t^\star)X_t(dH_t)
\end{aligned}$$

Since $H_t^\star H_t = I_m$, then $dH_t^\star H_t + H_t^\star dH_t + (dH_t^\star)(dH_t) = 0$. Hence, the matrix A defined by :

$$dA_t := H_t^\star dH_t + \frac{1}{2}(dH_t^\star)(dH_t),$$

is skew Hermitian so that all its entries are purely imaginary. The key point is to express dD_t by means of dA_t and $H_t^\star dX_tH_t$. Straightforward computations give :

$$\begin{aligned}
(dH_t^\star)(dH_t) &= (dA_t^\star H_t^\star)(H_tdA_t) = (dA_t^\star)(H_t^\star H_tdA_t) = -(dA_t)(dA_t), \\
H_t^\star(dX_t)(dH_t) &= H_t^\star(dX_t)H_t(dA_t) := d\Phi_t \\
(dH_t^\star)(dX_t)H_t &= -(dA_t)H_t^\star(dX_t)H_t = d\Phi_t^\star \\
(dH_t^\star)X_t(dH_t) &= (dA_t^\star)H_t^\star X_tH_t(dA_t) = (dA_t^\star)D_t(dA_t) := d\mu_t \\
dH_t &= H_t(dA_t + \frac{1}{2}(dA_t)(dA_t)).
\end{aligned}$$

Thus, setting $d\Gamma := (1/2)(dA)(dA)$, (17) writes

$$\begin{aligned}
dD_t = {} & H_t^\star dX_tH_t + (dA_t^\star D_t + D_tdA_t) + \frac{1}{2}((dA_t)(dA_t)D_t + D_t(dA_t)(dA_t)) \\
& + (dH_t^\star)(dX_t)H_t + H_t^\star(dX_t)(dH_t) + (dH_t^\star)X_t(dH_t) \\
= {} & H_t^\star dX_tH_t + (dA_t^\star D_t + D_tdA_t) + (d\Gamma_tD_t + D_td\Gamma_t) \\
& + (dH_t^\star)(dX_t)H_t + H_t^\star(dX_t)(dH_t) + (dH_t^\star)X_t(dH_t) \\
= {} & H_t^\star dX_tH_t + (D_tdA_t - dA_tD_t) + (d\Gamma_tD_t + D_td\Gamma_t) + d\Phi_t + d\Phi_t^\star + d\mu_t.
\end{aligned}$$

Writing down the diagonal terms, we get :

$$d\lambda_p(t) = \sum_{k,l} \overline{h_{kp}}(t)h_{lp}(t)dx_{kl}(t) + 2\lambda_p(t)d\gamma_{pp}(t) + 2\Re(d\phi_{pp}(t)) + d\mu_{pp}(t)$$

Hence, the local martingale part bracket equals to :

$$d\langle\lambda_p,\lambda_p\rangle_t = \left(2\sum_{k,l,r}\overline{h_{kp}}(t)h_{lp}(t)\overline{h_{lp}}(t)h_{rp}(t)x_{kr}(t) + 2\sum_{k,l,r}\overline{h_{kp}}(t)h_{lp}(t)\overline{h_{rp}}(t)h_{kp}(t)x_{lr}(t)\right)dt$$

$$= \left(2\sum_{l}h_{lp}(t)\overline{h_{lp}}(t)\sum_{k,r}\overline{h_{kp}}(t)h_{rp}(t)x_{kr}(t) + 2\sum_{k}h_{kp}(t)\overline{h_{kp}}(t)\sum_{l,r}\overline{h_{rp}}(t)h_{lp}(t)x_{lr}(t)\right)dt$$

$$= \left(2\lambda_p(t)\sum_{l}h_{lp}(t)\overline{h_{lp}}(t) + 2\lambda_p(t)\sum_{k}h_{kp}(t)\overline{h_{kp}}(t)\right) = 4\lambda_p(t)\,dt,$$

It follows that :

$$\text{Local martingale part of}\,(d\lambda_p(t)) = 2\sqrt{\lambda_p(t)}d\beta_p(t)$$

where β_p is a real Brownian motion. For the finite variation part, we start by evaluating the finite variation term of $\sum_{k,l}\overline{h_{kp}}h_{lp}dx_{kl}$ which will be denoted by dV.

$$dV = \text{Finite Variation of}\{\sum_{k,l}\overline{h_{kp}}(t)h_{lp}(t)dx_{kl}(t)\} = 2n\,dt\sum_{k,l}\overline{h_{kp}}(t)h_{kp}(t)\delta_{kl} = 2n\,dt.$$

The non-diagonal terms contribute :

$$(\lambda_m(t)-\lambda_j(t))da_{jm}(t) = \sum_{k,l}\overline{h_{kj}}(t)h_{lm}(t)dx_{kl}(t) + (\lambda_j(t)+\lambda_m(t))d\gamma_{jm}(t)$$
$$+ d\phi_{jm}(t) + \overline{d\phi_{mj}(t)} + d\mu_{jm}(t), \qquad t<\tau$$

Consequently, if $j\neq m$ and $r\neq s$, then :

$$(\lambda_m(t)-\lambda_j(t))(\lambda_s(t)-\lambda_r(t)) < da_{jm}(t), da_{rs}(t) >= 2\,\delta_{js}\delta_{rm}(\lambda_j(t)+\lambda_m(t))dt,$$

where δ is the Kroneker symbol. It follows that $d\Gamma$ and $d\mu$ are diagonal matrices. Besides,

$$d\gamma_{pp}(t) = \frac{1}{2}\sum_k\langle da_{pk}, da_{kp}\rangle_t = \frac{1}{2}\langle da_{pp}, da_{pp}\rangle_t - \sum_{k\neq p}\frac{\lambda_p(t)+\lambda_k(t)}{(\lambda_p(t)-\lambda_k(t))^2}dt$$

$$d\mu_{pp}(t) = -\sum_k\lambda_k\langle da_{pk}, da_{kp}\rangle_t = 2\sum_{k\neq p}\lambda_k(t)\frac{\lambda_p(t)+\lambda_k(t)}{(\lambda_p(t)-\lambda_k(t))^2}dt - \lambda_p(t)\langle da_{pp}, da_{pp}\rangle_t$$

$$\begin{aligned}
\Re(d\phi_{pp}(t)) &= \Re\left(\sum_{q,l}\sum_{k}\overline{h_{qp}}(t)h_{lk}(t)\langle dx_{ql}, da_{kp}\rangle_t\right)\\
&= \Re\left(2\sum_{q,l}\sum_{k\neq p}\overline{h_{qp}}(t)h_{lk}(t)\overline{h_{lk}}(t)h_{qp}(t)\frac{\lambda_p(t)+\lambda_k(t)}{\lambda_p(t)-\lambda_k(t)}dt + \langle (H^\star dXH)_{pp}, da_{pp}\rangle_t\right)\\
&= 2\sum_{k\neq p}\frac{\lambda_p(t)+\lambda_k(t)}{\lambda_p(t)-\lambda_k(t)}dt,
\end{aligned}$$

since $H_t^\star dX_t H_t$ is Hermitian and A_t is skew-Hermitian which ends the proof. ■

Corollary 3.1. *If $\lambda_1(0) > \cdots > \lambda_m(0)$, then the process $(U_t)_{t<\tau}$ defined by $U_t = 1/V(\lambda(t))$ is a local martingale.*

Proof : The statement above is equivalent to :

$$\sum_{i=1}^m x_i\frac{-(\partial_i^2 V)V^2 + 2(\partial_i V)^2 V}{V^4} \quad - \sum_{i=1}^m\left[n + \sum_{k\neq i}\frac{x_i+x_k}{x_i-x_k}\right]\frac{\partial_i V}{V^2} = 0,$$

that is :

$$\sum_{i=1}^m x_i\frac{-(\partial_i^2 V)V + 2(\partial_i V)^2}{V^2} \quad - \sum_{i=1}^m\left[n + \sum_{k\neq i}\frac{x_i+x_k}{x_i-x_k}\right]\frac{\partial_i V}{V} = 0,$$

In fact, we know that (cf [**19**])

$$(x_1,\ldots,x_m)\mapsto \log V(x_1,\ldots,x_m) = \sum_{i<j}\log(x_i-x_j)$$

is harmonic with respect to the infinitesimal generator of the Wishart eigenvalues process. This gives :

$$2\sum_{i=1}^m x_i\frac{(\partial_i^2 V)V - (\partial_i V)^2}{V^2} \quad + \sum_{i=1}^m\left[n + \sum_{k\neq i}\frac{x_i+x_k}{x_i-x_k}\right]\frac{\partial_i V}{V} = 0,$$

The proof ends since (cf [**43**] p. 216)

$$\sum_{i=1}^m x_i\partial_i^2 V = 0. \qquad ■$$

Remark. Another way of thinking is to use the following result due to König and O'Connell ([**78**]) : the eigenvalues process is the V-transform of the process consisting of m independent $BESQ(2(n-m+1)), n\geq m$. Thus, if G and $\hat{G}$ denote respectively the infinitesimal generators of these two processes, then $G(V)=0$ and forall C^2-function f

$$\hat{G}(f) = \frac{1}{V}G(Vf)$$

So, $\hat{G}(U) = \dfrac{1}{V}G(\mathbf{1}) = 0$ ([**41**]). We can also proceed as in the real case. Indeed, we are looking for C^2-functions $U(x_1, \cdots, x_m)$ such that :

$$\sum_{i=1}^{m} x_i \frac{\partial^2 U}{\partial x_i^2} \quad + \sum_{i=1}^{m}\left[n + \sum_{k\neq i}\frac{x_i + x_k}{x_i - x_k}\right]\frac{\partial U}{\partial x_i} = 0,$$

on the set $\{x_1 > \cdots > x_m \geq 0\}$. For $m = 2$, we get :

$$\frac{\partial U}{\partial x}\left[n + \frac{x+y}{x-y}\right] + \frac{\partial U}{\partial y}\left[n + \frac{x+y}{y-x}\right] + \left[x\frac{\partial^2 U}{\partial x^2} + y\frac{\partial^2 U}{\partial y^2}\right] = 0.$$

Hence setting :

$$z = x - y, \qquad U(x,y) = f(x-y) = f(z)$$

it follows that :

$$zf^{''}(z) + 2f^{'}(z) = 0.$$

Integrations yield :

$$zf^{'}(z) + f(z) = k, \quad k \in \mathbb{R} \quad \Rightarrow \quad f(z) = a + \frac{b}{z}, \quad z > 0,\, a, b \in \mathbb{R}$$

Hence, $U(x,y) = 1/(x-y)$ and one similarly shows that $U = 1/V$ in higher dimensions.■

Corollary 3.2. *If at time $t = 0$, the eigenvalues of X are distinct, then, they will never collide, i. e, $\tau = \infty$ a. s.*

Proof : this result follows from the fact that the continuous process U tends to infinity when $t \to \tau$ which is possible only if $\tau = \infty$ a. s. since every continuous local martingale is a time-changed Brownian motion (McKean argument, see [**68**]).

Remark. Using Theorem 3.1, it is easy to see that for $t < T_0 := \inf\{t, \det(X_t) = 0\}$ and $r \in \mathbb{R}$:

$$\begin{aligned}
d(\det(X_t)) &= 2\det(X_t)\sqrt{\operatorname{tr}(X_t^{-1})}d\nu_t + 2(n-m+1)\det(X_t)\operatorname{tr}(X_t^{-1})dt \\
d(\log(\det(X_t)) &= 2\sqrt{\operatorname{tr}(X_t^{-1})}d\nu_t + 2(n-m)\operatorname{tr}(X_t^{-1})dt, \\
d(\det(X_t)^r) &= 2(\det(X_t))^r\sqrt{\operatorname{tr}(X_t^{-1})}d\nu_t + 2r(n-m+r)(\det(X_t))^r\operatorname{tr}(X_t^{-1})dt
\end{aligned}$$

Lemma 3.1. *Take $X_0 \in \tilde{H_m^+}$. Then $X_t \in \tilde{H_m^+}$ for all $n \geq m$.*

Proof : For $n = m$, $\log\det(X)$ is a continuous local martingale and letting $r = m - n$, the same holds for $det(X)^{m-n}$. Both processes tend to infinity as $t \to T_0$. This can occur only if $T_0 = \infty$ a. s. by McKean argument. ■

3.1. Some skew-products. Let $(a\beta_t + \mu t)_{t\geq 0}$ be a Brownian motion with drift. We set $(K_t := e^{a\beta_t+\mu t})_{t\geq 0}$, then :

$$dK_t = aK_t d\beta_t + (\mu + \frac{a^2}{2})K_t dt$$

Define :

$$A_t = \int_0^t \mathrm{tr}(X_s^{-1})ds$$

Using this time-change, we can see that :

$$\begin{aligned} d(\det(X_{A_t})) &= 2\det(X_{A_t})dW_t + 2(n-m+1)\det(X_{A_t})dt \\ d(\log(\det(X_{A_t})) &= 2dW_t + 2(n-m)dt \end{aligned}$$

where W is a real Brownian motion. Hence, we deduce that :

$$\begin{aligned} (\det(X_{A_t}))_{t\geq 0} &\overset{\mathscr{L}}{=} \left(e^{2(\beta_t+(n-m)t)}\right)_{t\geq 0} \overset{\mathscr{L}}{=} \left(R^{(n-m)}_{\int_0^t e^{2(\beta_s+(n-m)s)}ds}\right)_{t\geq 0} \\ (\log(\det(X_{A_t})))_{t\geq 0} &\overset{\mathscr{L}}{=} (2(\beta_t + (n-m)t))_{t\geq 0} \end{aligned}$$

where $(R_t)_{t\geq 0}$ is a squared Bessel process of index $(n-m)$ (by Lamperti representation, [**80**]). We can also get a look at Brownian motions of ellipsoids already studied by Norris, Rogers and Williams (cf [**91**]), who considered the processes defined by :

$$F_t = G_t^T G_t \qquad \text{and} \qquad L_t = G_t G_t^T$$

where $(G_t)_{t\geq 0}$ is the right-invariant Brownian motion on $Gl(n,\mathbb{R})$ (right-invariant means that, for every $t, u \geq 0$, the law of $G_{t+u}G_t^{-1}$ does not depend on t and this right increment is independent of $\sigma(G_s,\, s \leq u)$). Let $(\gamma_i)_{1\leq i\leq m}$ denote the eigenvalues of F_t (or L_t), then

$$\frac{1}{2}d(\log\gamma_i(t)) = d\kappa_i(t) + \frac{1}{2}\sum_{k\neq i}\frac{\gamma_i(t)+\gamma_k(t)}{\gamma_i(t)-\gamma_k(t)}dt \qquad \forall t\geq 0, \qquad 1\leq i\leq m$$

where $(\kappa_i)_{1\leq i\leq m}$ are independent Brownian motions, which implies that :

$$\frac{1}{2}d\log(\det(F_t)) = \sqrt{m}N_t$$

where N is a real Brownian motion and since

$$\sum_{i=1}^m \sum_{k\neq i}\frac{\gamma_i+\gamma_k}{\gamma_i-\gamma_k} = 0$$

Then :

$$d(\det(F_t)) = 2\sqrt{m}\,\det(F_t)dN_t + 2m\,\det(F_t)dt$$

so that :

$$(\det(F_t))_{t\geq 0} \overset{\mathscr{L}}{=} \left(e^{2\sqrt{m}N_t}\right)_{t\geq 0} \overset{\mathscr{L}}{=} \left(\rho^{\sqrt{m},(0)}_{\int_0^t e^{2N_s}ds}\right)_{t\geq 0}$$

where ρ is a squared Bessel process of index 0.

3.2. Additivity Property.

PROPOSITION 3.1. *If* $(X_t)_{t\geq 0}$ *and* $(Y_t)_{t\geq 0}$ *are two independent Laguerre processes* $L(n,m,X_0)$ *and* $L(p,m,Y_0)$ *respectively, then* $(X_t+Y_t)_{t\geq 0}$ *is a* $L(n+p,m,X_0+Y_0)$*.*

Proof : Let us write $X_t = B_t^\star B_t$ and $Y_t = R_t^\star R_t$, where (B_t) and (R_t) are, respectively, $n\times m$ and $p \times m$ independent Brownian motions. Then $N_t = \begin{pmatrix} B_t \\ R_t \end{pmatrix}$ is a $(n+p)\times m$ complex Brownian matrix and we have

$$X_t + Y_t = B_t^\star B_t + R_t^\star R_t = N_t^\star N_t. \qquad \blacksquare$$

Now, we will introduce the Laguerre processes with noninteger dimension δ.

4. Laguerre Processes With Noninteger Dimension

Let $(X_t)_{t\geq 0}$ be a $L(n,m,X_0)$ with $n \geq m$. If $X_0 \in \tilde{H}_m^+$, and if $\sqrt{X_t}$ represent the symmetric square root of X_t, it is easy to show that the matrix O defined by $O_t := \sqrt{X_t}^{-1} B_t^\star$, where $X_t = B_t^\star B_t$, verifies $O^\star O = OO^\star = I_m$. Thus,

$$d\gamma_t = O_t dB_t = \sqrt{X_t}^{-1} B_t^\star dB_t$$

is a $m\times m$ complex Brownian matrix. Replacing in equation (14), we see that X_t is governed by the following SDE :

$$dX_t = \sqrt{X_t} d\gamma_t + d\gamma_t^\star \sqrt{X_t} + 2nI_m dt$$

THEOREM 4.1. *If* $(B_t)_{t\geq 0}$ *is a* $m\times m$ *complex Brownian matrix, then for every* $X_0 \in \tilde{H}_m^+$ *and* $\forall \delta \geq m$*, the SDE*

$$dX_t = \sqrt{X_t} dB_t + dB_t^\star \sqrt{X_t} + 2\delta I_m dt \tag{18}$$

has a unique strong solution in $\tilde{H}_m^+$*.*
Furthermore, if at time $t=0$ *the eigenvalues are distinct, then they satisfy the stochastic differential system :*

$$d\lambda_i(t) = 2\sqrt{\lambda_i(t)}\, d\beta_i(t) + 2\left[\delta + \sum_{k\neq i} \frac{\lambda_i(t)+\lambda_k(t)}{\lambda_i(t)-\lambda_k(t)}\right] dt \qquad 1 \leq i \leq m,$$

where the $(\beta_i)_{1\leq i\leq m}$ *are independent real Brownian motions.*

Proof : the proof of the second part of the Theorem is similar to that of integer dimensions. Thus $\det(X), \log\det(X)$ and $\det(X)^r$ satisfy the same SDE previously derived with δ instead of n. It remains only to prove the first part. Arguing as before, we claim that $T_0 = \infty$ a.s. Furthermore, the map $a \mapsto a^{1/2}$ is analytic in $\tilde{H}_m^+$ (see [**102**], p 134) so that the SDE has a unique strong solution defined on $t < T_0 = \infty$ a. s. $\blacksquare$

DEFINITION. Such a process is called *Laguerre process* of dimension δ, size m and starting from X_0. It will be denoted by $L(\delta, m, X_0)$.

REMARKS. 1/ Any process $(X_t)_{t\geq 0}$ solution of (18) is a diffusion whose infinitesimal generator is given by :

$$\mathscr{L} = 2\delta \ \mathrm{tr}(\Re(\frac{\partial}{\partial z})) + 2[\mathrm{tr}(x\Re(\frac{\partial}{\partial z})^2) + \mathrm{tr}(y\Im(\frac{\partial}{\partial z})^2)]$$

2/ A simple computation shows that for all i, j, k, $l \in \{1, \ldots, m\}$,

$$d\langle x_{ij}, x_{kl}\rangle_t = 2(x_{il}(t)\delta_{kj} + x_{kj}(t)\delta_{il})dt,$$

3/ In order to show that γ is a complex Brownian matrix, it suffices to write

$$O_t = S_t + iV_t, \qquad B_t = B_t^1 + iB_t^2,$$

which implies that :

$$d\gamma_t = (S_t dB_t^1 - V_t dB_t^2) + i(S_t dB_t^2 + V_t dB_t^1).$$

Then, from the fact that O is unitary, one has :

$$SS^T + VV^T = I_m, \quad VS^T - SV^T = 0.$$

from which we deduce that the real and imaginary parts are independent Brownian matrices using the Lévy characterisation of Brownian motion.

4.1. Some special functions. The infinitesimal generator of the Wishart eigenvalues process is given by ([**20**]) :

$$\begin{aligned}\mathscr{A} &= 2\sum_{i=1}^m \lambda_i \frac{\partial^2}{\partial \lambda_i^2} + \sum_{i=1}^m \left[\delta + \sum_{k\neq i} \frac{\lambda_i(t) + \lambda_k(t)}{\lambda_i(t) - \lambda_k(t)}\right] \frac{\partial}{\partial \lambda_i} \\ &= 2\left(\sum_{i=1}^m \lambda_i \frac{\partial^2}{\partial \lambda_i^2} + \frac{1}{2}\sum_{i=1}^m \left[\delta - m + 1 + 2\sum_{k\neq i} \frac{\lambda_i(t)}{\lambda_i(t) - \lambda_k(t)}\right] \frac{\partial}{\partial \lambda_i}\right).\end{aligned}$$

By the virtue of Eq. (36) p. 227 in [**89**], we deduce that : $g(X) = {}_0F_1^{(2)}(\delta/2, X)$ is an eigenfunction of $\mathscr{A}$ corresponding to the eigenvalue $2m$, that is $\mathscr{A} g = 2mg$. When $m = 1$, another eigenfunction is : $x \mapsto (\sqrt{x})^{1-b} K_{1-b}(2\sqrt{x})$, where K is the Macdonald function ([**85**]) and $b = \delta/2$. For Laguerre processes, the infinitesimal generator of $(\lambda_1, \ldots, \lambda_m)$ is given by :

$$\begin{aligned}\mathscr{A} &= 2\sum_{i=1}^m \lambda_i \frac{\partial^2}{\partial \lambda_i^2} + 2\sum_{i=1}^m \left[\delta + \sum_{k\neq i} \frac{\lambda_i(t) + \lambda_k(t)}{\lambda_i(t) - \lambda_k(t)}\right] \frac{\partial}{\partial \lambda_i} \\ &= 2\left(\sum_{i=1}^m \lambda_i \frac{\partial^2}{\partial \lambda_i^2} + \sum_{i=1}^m \left[\delta - m + 1 + \sum_{k\neq i} \frac{2\lambda_i(t)}{\lambda_i(t) - \lambda_k(t)}\right] \frac{\partial}{\partial \lambda_i}\right).\end{aligned}$$

Eq. 2. 13 p. 191 in [**31**] implies that ${}_0F_1^{(1)}(\delta, X)$ is an eigenfunction corresponding to the eigenvalue $2m$.

REMARK. Using relations between ${}_2F_1$, ${}_1F_1$ and ${}_0F_1$ and Th. XV. 3. 13 in [**52**], we can derive the differential system satisfied by ${}_0F_1$ in a more general setting (Jordan algebra). For complex Hermitian matrices, take $r = m$ and $d = 2$, for real symmetric matrices, take $r = m$ and $d = 1$).

The following subsection treats the existence and the uniqueness of a solution when $\delta > m - 1$ and $X_0 \in H_m^+$ (see [**19**] for the real case).

4.2. The Process X^+. If X is a Hermitian matrix, let X^+ be the Hermitian matrix $\max(X, 0)$. If we denote by (λ_i) the eigenvalues of X, then $(\lambda_i^+ = \max(\lambda_i, 0))$ are the eigenvalues of X^+ (see [**53**]).

THEOREM 4.2. *For all $\delta \in \mathbb{R}_+$ and $X_0 = x \in H_m$, the SDE*

$$dX_t = \sqrt{X_t^+}dB_t + dB_t^\star\sqrt{X_t^+} + 2\delta I_m dt$$

has a solution in H_m.

Proof : the mapping $a \mapsto \sqrt{a^+}$ is continuous on H_m, hence, by Th. 2. 3, p. 159 in [**68**], X exists up to its explosion time. Furthermore, from

$$||\sqrt{X^+}||^2 + ||\delta I||^2 \leq \delta^2 + ||\sqrt{X}||^2 \leq C(1 + ||X||^2),$$

it follows that this explosion time is infinite a. s. (cf [**68**] Th. 2. 4, page 163). ■

PROPOSITION 4.1. *If $\lambda_1(0) > \cdots > \lambda_m(0) \geq 0$, then, $\forall\, t < S := \inf\{t, \lambda_i = \lambda_j$ for some (i, j)$\}$, the eigenvalues of X^+ satisfy the following differential system :*

$$d\lambda_i(t) = 2\sqrt{\lambda_i^+(t)}d\nu_i(t) + 2\left(\delta + \sum_{k\neq i}\frac{\lambda_i^+(t) + \lambda_k^+(t)}{\lambda_i(t) - \lambda_k(t)}\right)dt, \qquad 1 \leq i \leq m,$$

Proof : this differential system can be shown in the same way as in Theorem 3.1, using :

$$d\langle x_{ij}, x_{kl}\rangle_t = 2(x_{il}^+(t)\delta_{kj} + x_{kj}^+(t)\delta_{il})dt.$$

COROLLARY 4.1. *$S = \infty$ p. s.*

Proof of the Corollary : following the lines of Bru's proof , then

$$U(\lambda_1(t), \ldots, \lambda_m(t)) = \frac{1}{\prod_{i<j}(\lambda_i(t) - \lambda_j(t))}$$

defines a local martingale. For instance, for $m = 2$, we are looking for C^2 functions U such that :

$$\frac{\partial U}{\partial x}\left[\delta + \frac{x^+ + y^+}{x - y}\right] + \frac{\partial U}{\partial y}\left[\delta + \frac{x^+ + y^+}{y - x}\right] + \left[x^+\frac{\partial^2 U}{\partial x^2} + y^+\frac{\partial^2 U}{\partial y^2}\right] = 0.$$

on the set $\{x > y\}$. Thus, setting

$$z = x - y, \qquad U(x, y) = f(x - y) = f(z)$$

we get :

$$zf^{''}(z) + 2f^{'}(z) = 0.$$

so that $U(x, y) = 1/(x - y)$ for $x > y$. However, by the virtue of Proposition 4.1 and the fact that, in the real case, $\log V$ is still a local martingale, the proof of Corollary 3.1 applies and the result follows. ■

PROPOSITION 4.2. *If $\lambda_1(0) > \cdots > \lambda_m(0) \geq 0$, then $\forall\, \delta > m - 1$, $\forall\, t > 0$, $\lambda_m(t) \geq 0$.*

Proof : the first step consists in showing that, if $\lambda_1 > \cdots > \lambda_{m-1}$ satisfy :

$$d\lambda_i(t) = 2\sqrt{\lambda_i^+(t)}\, d\nu_i(t) + 2\left(\delta + \sum_{k\neq i} \frac{\lambda_i^+(t) + \lambda_k^+(t)}{\lambda_i(t) - \lambda_k(t)}\right) dt, \qquad 1 \leq i \leq m-1,$$

then, the pathwise uniqueness holds for

$$d\lambda_m(t) = 2\sqrt{\lambda_m^+(t)}\, d\nu_m(t) + 2\left(\delta + \sum_{k\neq m} \frac{\lambda_m^+(t) + \lambda_k^+(t)}{\lambda_m(t) - \lambda_k(t)}\right) dt.$$

To do this, we use the Yamada-Watanabe criterion (see Th. IX. 3. 5 in [**101**]) :

$$|\sqrt{x^+} - \sqrt{y^+}|^2 \leq |x^+ - y^+| \leq |x - y|$$

$$\begin{aligned}
|b(t,x) - b(t,y)| &:= |\sum_{\lambda_k > x, \lambda_k > y} \left(\frac{x^+ + \lambda_k^+(t)}{x - \lambda_k(t)} - \frac{y^+ + \lambda_k^+(t)}{y - \lambda_k(t)}\right)| \\
&\leq \sum_{\lambda_k > x, \lambda_k > y} \left(\frac{|x^+ y - y^+ x|}{(\lambda_k(t) - x)(\lambda_k(t) - y)}\right) + \\
&\sum_{\lambda_k > x, \lambda_k > y} \left(\frac{\lambda_k(t)|x^+ - y^+|}{(\lambda_k(t) - x)(\lambda_k(t) - y)} + \frac{\lambda_k(t)^+|x - y|}{(\lambda_k(t) - x)(\lambda_k(t) - y)}\right) \\
&\leq \sum_{\lambda_k > x, \lambda_k > y} \left(\frac{|x - y|^2}{(\lambda_k(t) - x)(\lambda_k(t) - y)} + \frac{2|\lambda_k(t)||x - y|}{(\lambda_k(t) - x)(\lambda_k(t) - y)}\right),
\end{aligned}$$

since

$$\begin{aligned}
|xy^+ - yx^+| &= 0 \quad \text{if} \quad x > 0,\, y > 0 \quad \text{or} \qquad x < 0,\, y < 0, \\
&= -xy \qquad \text{else.}
\end{aligned}$$

Hence, if K is a compact set of $\mathbb{R}$ and if $x, y \in K$, then :

$$\begin{aligned}
|b(t,x) - b(t,y)| &\leq C|x - y| \sum_{\lambda_k > x, \lambda_k > y} \left(\frac{|\lambda_k(t)| + 1}{(\lambda_k(t) - x)(\lambda_k(t) - y)}\right) \\
&\leq C_1 \frac{1 + \max_{1\leq k\leq m-1} |\lambda_k|}{(\lambda_{m-1}(t) - x)(\lambda_{m-1}(t) - y)} |x - y|.
\end{aligned}$$

and the pathwise uniqueness holds using a suitable localisation. Next, for $x < \lambda_k(t)$ for all t, $1 \leq k \leq m-1$, we set :

$$g(t,x) = 2\left(\delta - \sum_{k=1}^{m-1} \frac{\lambda_k^+(t) + x}{\lambda_k(t) - x}\right) \mathbf{1}_{\{x>0\}} + 2(\delta - m + 1)\mathbf{1}_{\{x\leq 0\}},$$

g is a continuous function in both t and x. Besides,

$$g(t,\lambda_m(t)) = 2\left(\delta - \sum_{k=1}^{m-1} \frac{\lambda_k^+(t) + \lambda_m(t)}{\lambda_k(t) - \lambda_m(t)}\right) \mathbf{1}_{\{\lambda_m(t)\geq 0\}} + 2(\delta - m + 1)\mathbf{1}_{\{\lambda_m(t)<0\}},$$

Using a suitable localisation, we can define :

$$dY_t = 2\sqrt{Y_t^+}d\nu_m(t) + g(t,Y_t)dt, \qquad Y_0 = \lambda_m(0).$$

The second step consists in showing that $Y_t \geq 0$ a. s, $\forall t \geq 0$, which implies that :

$$g(t,Y_t) = 2\left(\delta - \sum_{k=1}^{m-1} \frac{\lambda_k^+(t) + Y_t}{\lambda_k(t) - Y_t}\right)$$

Together with the pathwise uniqueness give that $\lambda_m(t) = Y_t$ a. s, $\forall t \geq 0$. Let $T_a := \inf\{t, Y_t < a\}$ for fixed $a < 0$. We will prove that $T_a = \infty$ a. s. Let $T := \inf\{t \geq T_a,\, Y_t = 0\}$. On the set $\{T_a < \infty\}$, $Y_{T_a} = a$, so that $\forall t \in [T_a, T[$,

$$Y_t - a = Y_t - Y_{T_a} = \int_{T_a}^t g(s,Y_s)ds = 2(\delta - m + 1)(t - T_a) \geq 0,$$

since $\delta > m-1$. Consequently, $\forall t \in [T_a, T[$, $Y_t \geq a$, which is in contradiction with the definition of T_a. ■

Theorem 4.3. *If $\lambda_1(0) > \cdots > \lambda_m(0) \geq 0$, then, for all $\delta > m-1$,* (18) *has a unique solution in H_m^+ in the sense of probability law.*

Proof : by Proposition 4.2, the solution of the SDE in Th. 4.2 remains positive for all $t > 0$, thus, it is a solution of (18). ■

Theorem 4.4. *Whenever the SDE (18) have a solution $(X_t)_{t\geq 0}$ in H_m^+, its distribution is given by its Laplace transform :*

(19) $\qquad \mathbb{E}_{X_0}(\exp-(\mathrm{tr}\, uX_t)) = (\det(I_m + 2tu))^{-\delta} \exp(-\mathrm{tr}(X_0(I_m + 2tu)^{-1} u)),$

for all u in H_m^+.

Proof : for $s \in H_m^+$, set $g(t,s) = \Delta_t^{-\delta} \exp(-V(t,s))$, where

$$\Delta_t = \det(I_m + 2ut), \quad W_t = (I_m + 2ut)^{-1}u, \quad V(t,s) = \mathrm{tr}(sW_t),$$

Note first that $W \in H_m$. Next, we need a Lemma :

Lemma 4.1. *g satisfies the heat equation : $\mathscr{L}g = \dfrac{\partial g}{\partial t}$ where $\mathscr{L}$ is the infinitesimal generator of X.*

Proof of the Lemma : Writing $s = x + iy$, and since x is symmetric, y skew-symmetric and W is Hermitian, we can see that $\mathrm{tr}(sW_t) = \mathrm{tr}(xM + iyN)$ where

$$M = \frac{W + \overline{W}}{2} \qquad N = \frac{W - \overline{W}}{2}.$$

Since $M^T = M$ and $N^T = -N$, we deduce that g satisfies conditions of Proposition 2.1. Furthermore, one can see that :

$$\begin{aligned}
\frac{\partial g}{\partial t} &= -g(2\delta\ \mathrm{tr}(W_t) - 2\,\mathrm{tr}(sW_t^2)), \\
\frac{\partial g}{\partial x_{pq}} &= -g\frac{\partial V}{\partial x_{pq}}, \\
\frac{\partial g}{\partial y_{pq}} &= -g\frac{\partial V}{\partial y_{pq}}, \\
\frac{\partial^2 g}{\partial x_{pq}x_{kj}} &= g\frac{\partial V}{\partial x_{pq}}\frac{\partial V}{\partial x_{kj}}, \\
\frac{\partial^2 g}{\partial y_{pq}y_{kj}} &= g\frac{\partial V}{\partial y_{pq}}\frac{\partial V}{\partial y_{kj}},
\end{aligned}$$

which implie that :

$$\begin{aligned}
\frac{\partial g}{\partial x} &= -g\frac{\partial V}{\partial x} = -gM, \\
\frac{\partial g}{\partial y} &= -g\frac{\partial V}{\partial y} = igN, \\
\frac{\partial^2 g}{\partial x^2} &= g\left(\frac{\partial V}{\partial x}\right)^2 = gM^2, \\
\frac{\partial^2 g}{\partial y^2} &= g\left(\frac{\partial V}{\partial y}\right)^2 = -gN^2, \\
\frac{\partial^2 g}{\partial x\partial y} &= -igMN, \\
\frac{\partial^2 g}{\partial y\partial x} &= -igNM
\end{aligned}$$

Second order derivatives give

$$\begin{aligned}
\mathrm{tr}(y\left(\frac{\partial^2 g}{\partial x\partial y} + \frac{\partial^2 g}{\partial y\partial x}\right)) &= -ig\ \mathrm{tr}(yW^2), \\
\mathrm{tr}(x\left(\frac{\partial^2 g}{\partial x^2} - \frac{\partial^2 g}{\partial y^2}\right)) &= g\ \mathrm{tr}(x(M^2 + N^2)) = g\ \mathrm{tr}(xW^2).
\end{aligned}$$

The result follows since $\mathrm{tr}(M) = \mathrm{tr}(W)$. ■
Now, let us consider the process $(Z(t, X_t))$ defined by :

$$Z(t, X_t) = g(t_1 - t, X_t), \qquad \forall\, t \le t_1$$

for a fixed t_1. Z is a bounded local martingale by Lemma 4.1 and thus is a martingale. The result follows from a simple use of the optional stopping theorem. ■

REMARK. From the Laplace transform, we easily deduce the additivity property for fixed t.

COROLLARY 4.2. *Let $(X_t)_{t\ge 0}$ be a Laguerre process $L(\delta, m, x)$ where $x \in \tilde{H_m}^+$. For $\delta > m-1$, the semi-group of $(X_t)_{t\ge 0}$ is given by the following density :*

$$p_t^\delta(x, y) = \frac{1}{(2t)^{m\delta}\Gamma_m(\delta)} \exp -(\frac{1}{2t}\,\mathrm{tr}(x+y))\,(det\, y)^{\delta-m}{}_0F_1^{(1)}(\delta; \frac{xy}{4t^2})\mathbf{1}_{\{y>0\}}$$

with respect to Lebesgue measure $dy = \prod_{p\le q} dy^1_{pq} \prod_{p<q} dy^2_{pq}$ where $y = y^1 + iy^2$.

Proof : this result can be easily deduced from the integer case for which X_t is a non-central complex Wishart variable $W(n, 2tI_m, x)$ (cf [**69**]) with density given by :

$$f_t(x, y) = \frac{1}{(2t)^{mn}\Gamma_m(n)} \exp -(\frac{1}{2t}\,\mathrm{tr}(x+y))\,(det\, y)^{n-m}{}_0F_1^{(1)}(n; \frac{xy}{4t^2})\mathbf{1}_{\{y>0\}}$$

with respect to dy. Hence, writing δ instead of n and denoting by W_t this new variable (starting from x), we can see that : ($|y|$ denotes $\det(y)$)

$$\begin{aligned}
E_x(e^{-\mathrm{tr}\, uW_t}) &= \frac{1}{(2t)^{m\delta}\Gamma_m(\delta)} e^{-\frac{\mathrm{tr}\, x}{2t}} \int_{y>0} \exp\left(-\frac{1}{2t}\,\mathrm{tr}((I+2ut)y)\right) |y|^{\delta-m}{}_0F_1^{(1)}(\delta; \frac{xy}{4t^2})dy \\
&= \frac{2t^{m\delta}|x|^{-\delta}}{\Gamma_m(\delta)} e^{-\frac{\mathrm{tr}\, x}{2t}} \int_{z>0} \exp(-2t\ \mathrm{tr}(x^{-\frac{1}{2}}(I+2ut)x^{-\frac{1}{2}}z)|z|^{\delta-m}{}_0F_1^{(1)}(\delta; z)dz \\
&= \exp(-\frac{\mathrm{tr}\, x}{2t})|I+2ut|^{-\delta} \exp\left(\mathrm{tr}(\frac{x}{2t}(I+2ut)^{-1})\right) \\
&= |I+2ut|^{-\delta} \exp\left(-\frac{1}{2t}\,\mathrm{tr}(x(I+2ut)^{-1}(I+2ut-I))\right) \\
&= |I+2ut|^{-\delta} \exp\left(-\,\mathrm{tr}(x(I+2ut)^{-1}u)\right)
\end{aligned}$$

which is equal to (19). ■

REMARKS. 1/ In the last proof, we used the change of variables $z = x^{1/2}yx^{1/2}$ which gives $dz = |x|^m dy$. For the second integral, see [**52**], Proposition XV.1.3, p 319.
2/ The expression of the semi-group extends continuously to the degenerate case :

$$p_t^\delta(0_m, y) = \frac{1}{(2t)^{m\delta}\Gamma_m(\delta)} \exp -(\frac{\mathrm{tr}(y)}{2t})\,(det\, y)^{\delta-m}\mathbf{1}_{\{y>0\}}$$

where 0_m denotes the null matrix.

Corollary 4.3. *For $\delta > m-1$, the semi-group of the eigenvalues process is given by :*

$$q_t(x,y) = \frac{V(y)}{V(x)} det \left(\frac{1}{2t} \left(\frac{y_j}{x_i} \right)^{\nu/2} e^{-\frac{(x_i+y_j)}{2t}} I_\nu(\frac{\sqrt{x_i y_j}}{t}) \right)$$

where $\delta = m+\nu$, $\nu > -1$, $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m)$ such that $x_1 > \dots > x_m > 0$, $y_1 > \dots > y_m > 0$.

Proof: the expression of the semi-group can be computed using Karlin-MacGregor formula ([**72**]) since, for $\delta > m-1$, the eigenvalues process is the V-transform of the process obtained from m independent $BESQ(2(\delta - m + 1))$ conditioned never to collide (cf [**78**]). In fact, we can extend König and O'connell result for any $\delta > m-1$.
Another proof is given by Péché (cf [**94**], p. 68). Here, we will deduce the expression of $q_t(x,y)$ from $p_t(x,y)$ following Muirhead (cf [**89**]), namely, by projection on the unitary group : first, we will use the Weyl integration formula, then give a determinantal representation of hypergeometric functions of two Hermitian matrix arguments. Let us state the Weyl integration formula (cf [**52**]) in the complex case : for any Borel function f,

$$\int_{H_m} f(A)dA = C_m \int_{U(m)} \int_{\mathbb{R}^m} f(uau^\star)\alpha(du)(V(a))^2 da_1 \dots da_m,$$

where $C_m = \dfrac{\pi^{m(m-1)}}{\Gamma_m(m)}$, $U(m)$ is the unitary group, α is the normalized Haar measure on $U(m)$, $a = diag(a_i)$ and $A = uau^\star$. Hence, the semi-group of the eigenvalues process is given by ([**69**]) :

$$\begin{aligned} q_t(x,y) &= C_m V(y^2) \int_{U(m)} p_t(\tilde{x}, u\tilde{y}u^\star)\alpha(du) \\ &= \frac{C_m (V(y)^2)}{(2t)^{m\delta}\Gamma_m(\delta)} \prod_{i,j=1}^m e^{-\frac{x_i+y_j}{2t}} \left(\prod_{i=1}^m y_j \right)^{\delta - m} \int_{U(m)} {}_0F_1^{(1)}(\delta; \frac{\tilde{x}u\tilde{y}u\star}{4t^2})\alpha(du) \\ &= \frac{\pi^{m(m-1)}(V(y)^2)}{(2t)^{m(m+\nu)}\Gamma_m(m)\Gamma_m(m+\nu)} \prod_{i,j=1}^m e^{-\frac{x_i+y_j}{2t}} \left(\prod_{i=1}^m y_j \right)^{\nu} {}_0F_1^{(1)}(m+\nu; \frac{\tilde{x}}{4t^2}; \tilde{y}), \end{aligned}$$

where $\tilde{y} = diag(y_j)$, x is a positive definite matrix with eigenvalues $x_1, \dots, x_m$, $\delta = m+\nu, \nu > -1$. (For the last equality, see [**69**]). Next, we need a lemma.

LEMMA 4.2. *Let $B, C \in H_m$ and let (b_i), (c_i) be respectively their eigenvalues. Then,*

$$ {}_pF_q^{(1)}((m+\mu_i)_{1\le i\le p},(m+\phi_j)_{1\le j\le q};B,C) = \pi^{\frac{m(m-1)}{2}(p-q-1)}\Gamma_m(m)\prod_{i=1}^{p}\frac{(\Gamma(\mu_i+1))^m}{\Gamma_m(m+\mu_i)} $$
$$ \prod_{j=1}^{q}\frac{\Gamma_m(m+\phi_j)}{(\Gamma(\phi_j+1))^m}\quad\frac{\det\left({}_p\mathscr{F}_q((\mu_i+1)_{1\le i\le p},(1+\phi_j)_{1\le j\le q};b_lc_f\right)_{l,f}}{h(B)h(C)} $$

$\forall\, \mu_i, \phi_j > -1, 1 \le i \le p,\ 1 \le j \le q.$

Proof : Recall that the hypergeometric function of two matrix arguments is given by the following series :

$$ {}_pF_q^{(1)}((a_i)_{1\le i\le p},(e_j)_{1\le j\le q};B,C) = \sum_{k=0}^{\infty}\sum_{\tau}\frac{\prod_{i=1}^{p}(a_i)_\tau}{\prod_{j=1}^{q}(e_j)_\tau}\frac{J_\tau^{(1)}(B)J_\tau^{(1)}(C)}{J_\tau^{(1)}(1_m)k!}, $$

It is well known that :

$$ J_\tau^{(1)}(B) = \frac{k!d_\tau}{(m)_\tau}s_\tau(b_1,\dots,b_m), $$

where s_τ is the Schur function and $d_\tau = s_\tau(I)$ is the representation trace or degree (cf [**62**] or [**52**]). The hypergeometric series of two matrix arguments is written :

$$ {}_pF_q^{(1)}((m+\mu_i)_{1\le i\le p},(m+\phi_j)_{1\le j\le q};B,C) = \sum_{k=0}^{\infty}\sum_{\tau}\frac{\prod_{i=1}^{p}(m+\mu_i)_\tau}{\prod_{j=1}^{q}(m+\phi_j)_\tau}\frac{s_\tau(B)s_\tau(C)}{(m)_\tau}, $$

Now, we write :

$$ \begin{aligned} (m+\mu_i)_\tau &= \prod_{r=1}^{m}\frac{\Gamma(\mu_i+m+k_r-r+1)}{\Gamma(\mu_i+m-r+1)} = \prod_{r=1}^{m}\frac{\Gamma(\mu_i+1+k_r+\delta_r)}{\Gamma(\mu_i+m-r+1)} \\ &= \pi^{m(m-1)/2}\frac{(\Gamma(\mu_i+1))^m}{\Gamma_m(m+\mu_i)}\prod_{r=1}^{m}(\mu_i+1)_{k_r+\delta_r} \end{aligned} $$

where $\delta_r = m-r$. Doing the same thing for each $(m+\phi_j)_\tau$ and for $(m)_\tau$ give :

$$ {}_pF_q^{(1)}((m+\mu_i)_{1\le i\le p},(m+\phi_j)_{1\le j\le q};B,C) = \pi^{\beta}\Gamma_m(m)\prod_{i=1}^{p}\frac{(\Gamma(\mu_i+1))^m}{\Gamma_m(m+\mu_i)}\prod_{j=1}^{q}\frac{(\Gamma(\phi_j+1))^m}{\Gamma_m(m+\phi_j)} $$
$$ \sum_{k=0}^{\infty}\sum_{\tau}\prod_{r=1}^{m}\left(\frac{\prod_{i=1}^{p}(\mu_i+1)_{k_r+\delta_r}}{\prod_{j=1}^{q}(\phi_j+1)_{k_r+\delta_r}}\right)\quad\frac{s_\tau(B)s_\tau(C)}{\prod_{r=1}^{m}(1)_{k_r+\delta_r}} $$

where $\beta = m(m-1)(p-q-1)/2$. The Lemma results from *Hua formula* (cf [**51**]) :

LEMMA 4.3. *Given an entire function f, i.e, $f(z) = \sum_{k=0}^{\infty} e_k z^k$, then*

$$\frac{\det(f(b_i c_j))_{i,j}}{h(B)h(C)} = \sum_{k=0}^{\infty} \sum_{\tau} \left(\prod_{r=1}^{m} e_{k_r+\delta_r} \right) \frac{s_\tau(B) s_\tau(C)}{s_\tau(I_m)}.$$

Thus

$$_pF_q^{(1)}((m+\mu_i)_{1\le i\le p},(m+\phi_j)_{1\le j\le q}; B, C) = \pi^{\frac{m(m-1)}{2}(p-q-1)} \Gamma_m(m) \prod_{i=1}^{p} \frac{\Gamma(\mu_i+1)}{\Gamma_m(m+\mu_i)}$$

$$\prod_{j=1}^{q} \frac{\Gamma_m(m+\phi_j)}{\Gamma(\phi_j+1)} \quad \frac{\det\left(\sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}(\mu_i+1)_k}{\prod_{j=1}^{q}(\phi_j+1)_k} \frac{(b_l c_p)^k}{k!}\right)_{l,p}}{h(B)h(C)}$$

$$= \pi^{\frac{m(m-1)}{2}(p-q-1)} \Gamma_m(m) \prod_{i=1}^{p} \frac{\Gamma(\mu_i+1)}{\Gamma_m(m+\mu_i)} \prod_{j=1}^{q} \frac{\Gamma_m(m+\phi_j)}{\Gamma(\phi_j+1)}$$

$$\frac{\det\left({}_p\mathscr{F}_q((\mu_i+1)_{1\le i\le p},(1+\phi_j)_{1\le j\le q}; b_l c_f\right)_{l,f}}{h(B)h(C)} \qquad \blacksquare$$

REMARK. For $p = 0$ and $q \geq 1$, we have :

$$_0F_q^{(1)}((m+\phi_j)_{1\le j\le q}; B, C) = \pi^{-\frac{m(m-1)}{2}(q+1)} \Gamma_m(m) \prod_{j=1}^{q} \frac{\Gamma_m(m+\phi_j)}{(\Gamma(\phi_j+1))^m}$$

$$\frac{\det\left({}_0\mathscr{F}_q((1+\phi_j)_{1\le j\le q}; b_l c_f\right)_{l,f}}{h(B)h(C)},$$

and similarly,

$$_0F_0^{(1)}(B, C) = \frac{\Gamma_m(m)}{\pi^{\frac{m(m-1)}{2}}} \frac{\det(e^{b_l c_f})_{l,f}}{h(B)h(C)}$$

which can be viewed as Harish-Chandra formula for the "Itzykson-Zuber" integral ([**34**]).

We now proceed to the end of the proof. Taking $p = 0$, $q = 1$, $B = \tilde{x}/(4t^2)$, $C = \tilde{y}$, we get :

$$_0F_1^{(1)}(m+\nu; \frac{\tilde{x}}{4t^2}, \tilde{y}) = \frac{(4t^2)^{m(m-1)/2} \Gamma_m(m+\nu) \Gamma_m(m)}{\pi^{m(m-1)} (\Gamma(\nu+1))^m} \frac{\det\left({}_0\mathscr{F}_1((\nu+1); \frac{x_i y_j}{4t^2})\right)}{h(x)h(y)}$$

The expression of $q_t(x, y)$ follows from a simple computation and from the fact that :

$$\frac{{}_0\mathscr{F}_1((\nu+1); x_i y_j/4t^2))_{i,j}}{\Gamma(\nu+1)} = \left(\frac{2t}{\sqrt{x_i y_j}}\right)^{\nu} I_\nu\left(\frac{\sqrt{x_i y_j}}{t}\right) \qquad \blacksquare$$

PROPOSITION 4.3. *The measure defined by $\rho(dx) = (\det(x))^{\delta-m}\, dx$ on $\tilde{H}_m^{+}$ is invariant under the semi-group, i. e, $\rho P_t = \rho$.*

Proof : Denote by P_t the semi-group of a Laguerre process $L(\delta, m, x)$ with $\delta > m-1$. Then, we have to show that :

$$\int_{x>0} P_t f(x)\rho(dx) = \int_{y>0} f(y)\rho(dy) \qquad \forall\, f \in C_0(\tilde{H_m}^+).$$

Indeed, by Corollary 4.2, it follows that

$$\int_{x>0} P_t f(x)\rho(dx) = \frac{1}{(2t)^{m\delta}\Gamma_m(m)} \int_{y>0} e^{-\operatorname{tr}(\frac{y}{2t})}|y|^{\delta-m} f(y) \left(\int_{x>0} e^{-\operatorname{tr}(\frac{x}{2t})}|x|^{\delta-m} {}_0F_1^{(1)}(\delta, \frac{xy}{4t^2})dx\right) dy$$

$$= \frac{(2t)^{m\delta}}{\Gamma_m(m)} \int_{y>0} e^{-\operatorname{tr}(\frac{y}{2t})}|y|^{-m} f(y) \left(\int_{x>0} e^{-\operatorname{tr}(2ty^{-1/2}zy^{-1/2})}|z|^{\delta-m} {}_0F_1^{(1)}(\delta, z)dz\right) dy$$

$$= \int_{y>0} f(y) e^{-\operatorname{tr}(\frac{y}{2t})}|y|^{\delta-m} {}_1F_1^{(1)}(\delta,\delta, \frac{y}{2t})dy$$

$$= \int_{y>0} f(y) e^{-\operatorname{tr}(\frac{y}{2t})}|y|^{\delta-m} e^{\operatorname{tr}(\frac{y}{2t})}dy = \int_{y>0} f(y)\rho(dy)$$

where we used the same change of variables as in the proof of Corollary 4.2 and Proposition XV. 1. 3, p 319 in [**52**] (see remark below the proof). ■

Remark. For Wishart processes, it is easy to see that $\mu(dx) := (\det(x))^{\frac{\delta}{2}-\frac{m+1}{2}}\mathbf{1}_{\{x>0\}}dx$ is invariant under the semi-group.

4.3. Some Orthogonal Polynomials.

Proposition 4.4. *Let $L(m+\nu, m, X_0)$, $\nu > -1$, be a Laguerre process and let $Y = (\lambda_1, \dots, \lambda_m)$ be its eigenvalues process starting at $x \in \mathbb{R}^m$. Set :*

$$Z_t := e^{2\beta t}\, Y(\frac{1-e^{-2\beta t}}{2\beta}) := e^{2\beta t}\, Y_{A_t} \qquad \beta \in \mathbb{R}^+,$$

then, the semi-group of $(Z_t)_{t\geq 0}$ is given by :

$$\tilde{q}_t(x,y) = (r\beta)^{m(m+\nu)} e^{-\operatorname{tr}(\beta x)} W(y) \sum_{k\geq 0}\sum_{\tau} \frac{L_\tau^\nu(\beta x) L_\tau^\nu(\beta y)}{\mathcal{N}_\tau} e^{-2\beta|\tau|t}, \qquad y \in \mathbb{R}^m$$

*where , $W(y) = (\det(y))^\nu\,(V(y))^2$, τ is a partition of k, L_τ^ν is the generalized complex Laguerre polynomial ([**?**], see also [**89**] for the real case, [**5**], [**52**] for a general setting), $\mathcal{N}_\tau$ is the normalisation constant given by :*

$$\mathcal{N}_\tau \quad = \quad \int_{[0,\infty[^m} (L_\tau^\nu(x))^2\,(\det(y))^\nu e^{-\operatorname{tr}(y)}(V(y))^2 dy,$$

Proof : Recall that the semi group of $(Y_t)_{t\geq 0}$ is given by :

$$q_t(x,y) = \frac{\pi^{m(m-1)}(V(y)^2)(\det(y))^\nu}{(2t)^{m(m+\nu)}\Gamma_m(m)\Gamma_m(m+\nu)} e^{-\frac{\operatorname{tr}(x+y)}{2t}}\, {}_0F_1^{(1)}(m+\nu; \frac{x}{4t^2}; y)$$

Using the definition of Z and setting $r = e^{-2\beta t}$, we have

$$\begin{aligned}\tilde{q}_t(x,y) &= e^{-2\beta t m} q_{A_t}(x, e^{-2\beta t} y)\\ &= C_m(y) \left(\frac{r\beta}{(1-r)}\right)^{m(m+\nu)} \exp\left(-\beta \operatorname{tr}(\frac{x}{1-r} + \frac{ry}{1-r})\right) {}_0F_1^{(1)}(m+\nu; \frac{\beta x}{1-r}; \frac{r\beta y}{1-r})\\ &= C_m(y) e^{-\operatorname{tr}(\beta x)} \left(\frac{r\beta}{(1-r)}\right)^{m(m+\nu)} \exp\left(-\frac{r\beta}{1-r}\operatorname{tr}(x+y)\right) {}_0F_1^{(1)}(m+\nu; \frac{\beta x}{1-r}; \frac{r\beta y}{1-r})\end{aligned}$$

where $C_m(y) = \dfrac{\pi^{m(m-1)}(h(y)^2)(\det(y))^\nu}{\Gamma_m(m)\Gamma_m(m+\nu)}$. Then, we use the generating function ([**5**], p 201) :

$$\sum_{k\geq 0}\sum_{\tau} \frac{L_\tau^\nu(u) L_\tau^\nu(v)}{\mathscr{N}_\tau} r^{|\tau|} = \frac{\pi^{m(m-1)}}{\Gamma_m(m)\Gamma_m(m+\nu)} \left(\frac{1}{(1-r)}\right)^{m(m+\nu)} \exp\left(-\frac{r}{1-r}\operatorname{tr}(u+v)\right)$$
$${}_0F_1^{(1)}(m+\nu; \frac{u}{1-r}; \frac{rv}{1-r}), \qquad |r| < 1$$

An easy computation gives the result. ■

Remarks. In the univariate case $m = 1$, Z is a squared Ornstein-Ulhenbeck process ([**96**]). Its semi-group density is given by a similar bilinear series ([**114**]). For the matrix case, define $(R_t)_{t\geq 0}$ as the unique strong solution of :

$$dR_t = d\gamma_t + \beta R_t dt, \qquad R_0 = \gamma_0$$

where γ is a $n \times m$ complex matrix Brownian motion and $\beta \in \mathbb{R}^+$. Then, we consider $S_t := R_t^\star R_t$. The process $(S_t)_{t\geq 0}$ satisfies (using Itô's formula and that $(dB_t := \sqrt{S_t}^{-1} R_t^\star d\gamma_t)_{t\geq 0}$ defines a complex matrix Brownian motion) :

$$dS_t = \sqrt{S_t} dB_t + dB_t^\star \sqrt{S_t} + (2\beta S_t + 2n I_m)dt, \qquad S_0 = \gamma_0^\star \gamma_0$$

Thus, if $(X_t = B_t^\star B_t)_{t\geq 0}$ is $L(n,m,x)$, then we can easily see that :

$$\left(S_t \overset{\mathscr{L}}{=} e^{2\beta t} X_{A_t}, t \geq 0\right)$$

Using Theorem 4.1 and Theorem 4.3, the following SDE :

$$dS_t = \sqrt{S_t} dB_t + dB_t^\star \sqrt{S_t} + (2\beta S_t + 2\delta I_m)dt, \qquad S_0 = s$$

has a unique strong solution. Such a process is the complex analog of the matrix squared Ornstein-Ulhenbeck process already defined in [**19**], and $(Z_t)_{t\geq 0}$ is its eigenvalues process.

5. Girsanov Formula and Absolute-continuity Relations

The index $\nu > -1$ of a $L(\delta, m, x)$ is defined by $\nu = \delta - m$. In this section, we will discuss in the same way as in [**40**] to derive absolute-continuity relations between different indices.

5.1. Positive Indices. Take a matrix-valued Hermitian predictable process H. Let Q_x^δ be the probability law of $L(\delta, m, x)$ for $\delta > m-1$ and $x \in \tilde{H_m^+}$. Define :

$$\begin{aligned} L_t &= \int_0^t \frac{\mathrm{tr}(H_s dB_s + \overline{H_s}\,\overline{dB_s})}{2}, \\ \Phi_t &= \exp\left(L_t - \frac{1}{2}\int_0^t \mathrm{tr}(H_s^2) ds\right), \end{aligned}$$

where B is a complex Brownian matrix under Q_x^δ. We can easily see that the process β defined by $\beta_t = B_t - \int_0^t H_s ds$ is a Brownian matrix under the probability

$$\mathbb{P}_x^H|_{\mathscr{F}_t} := \Phi_t \cdot Q_x^\delta|_{\mathscr{F}_t},$$

Furthermore, $(X_t)_{t\geq 0}$ is a solution of

$$dX_t = \sqrt{X_t} d\beta_t + d\beta_t^\star \sqrt{X_t} + (\sqrt{X_t} H_t + H_t\sqrt{X_t} + 2\delta I_m)dt. \tag{20}$$

For $H_t = \nu\sqrt{X_t}^{-1}$, (20) becomes

$$dX_t = \sqrt{X_t} d\beta_t + d\beta_t^\star \sqrt{X_t} + 2(\delta+\nu) I_m dt,$$

so that $(X_t)_{t\geq 0}$ is a $L(\delta+\nu, m, x)$ under $\mathbb{P}_x^H$. Thus, we proved that :

THEOREM 5.1. *For $\delta > m-1$,*

$$Q_x^{\delta+\nu}|_{\mathscr{F}_t} = \exp\left(\frac{\nu}{2}\int_0^t \mathrm{tr}(\sqrt{X_s}^{-1} dB_s + \overline{\sqrt{X_s}^{-1} dB_s}) - \frac{\nu^2}{2}\int_0^t \mathrm{tr}({X_s}^{-1}) ds\right) \cdot Q_x^\delta|_{\mathscr{F}_t}. \tag{21}$$

PROPOSITION 5.1.

$$Q_x^{m+\nu}|_{\mathscr{F}_t} = \left(\frac{det(X_t)}{\det(x)}\right)^{\nu/2} \exp\left(-\frac{\nu^2}{2}\int_0^t \mathrm{tr}({X_s}^{-1}) ds\right) \cdot Q_x^m|_{\mathscr{F}_t}. \tag{22}$$

Proof : We know that $\nabla_u(\det(u)) = \det(u) u^{-1}$, hence, $\nabla_u(\log(\det(u))) = u^{-1}$. Then, using the fact that for $\delta = m$, $\log(\det(X))$ is a local martingale,

$$\begin{aligned} \log(\det(X_t)) &= \log(\det(X_0)) + \int_0^t \mathrm{tr}({X_s}^{-1}(\sqrt{X_s} dB_s + dB_s^\star \sqrt{X_s})) \\ &= \log(\det(X_0)) + \int_0^t \mathrm{tr}(\sqrt{X_s}^{-1} dB_s + \overline{\sqrt{X_s}^{-1} dB_s}). \end{aligned}$$

■

From (22), it follows that :

COROLLARY 5.1.

$$\begin{aligned} Q_x^m(\exp\left(-\frac{\nu^2}{2}\int_0^t \mathrm{tr}(X_s^{-1}) ds\right) | X_t = y) &= \frac{\det(y)}{\det(x)}^{-\nu/2} \frac{p_t^{m+\nu}(x,y)}{p_t^m(x,y)} \\ &= \frac{\Gamma_m(m)}{\Gamma_m(m+\nu)} (\det(z))^{\nu/2} \frac{{}_0F_1^{(1)}(m+\nu, z)}{{}_0F_1^{(1)}(m, z)} := \frac{\tilde{I}_\nu(z)}{\tilde{I}_0(z)}, \end{aligned}$$

where $z = xy/(4t^2)$.

COROLLARY 5.2. *Let* X *be a Laguerre process* $L(m,m,x)$*, then, as* $t \to \infty$ *:*

$$\frac{4}{(m\log t)^2}\int_0^t \mathrm{tr}(X_s)^{-1}ds \qquad \overset{\mathscr{L}}{\to} \qquad T_1(\beta)$$

where T_1 *is the first hitting time of* 1 *by a standard Brownian motion* β*.*

Proof : On one hand :

$$Q_x^m(\exp\left(-\frac{2\nu^2}{(m\log t)^2}\int_0^t \mathrm{tr}(X_s^{-1})ds\right)|X_t = ty) = \frac{\Gamma_m(m)}{\Gamma_m(m+2\nu/m\log t)}(\det(xy/4t))^{\nu/m\log t}$$
$$\frac{{}_0F_1^{(1)}(m+2\nu/m\log t, xy/4t)}{{}_0F_1^{(1)}(m, xy/4t)}.$$

Noting that $(t^m)^{-\nu/m\log t} = e^{-\nu}$, and since both hypergeometric functions converge to 1 as $t \to \infty$, we obtain :

$$Q_x^m\left(\exp(-\frac{2\nu^2}{(m\log t)^2}\int_0^t \mathrm{tr}(X_s^{-1})ds|X_t = ty\right) \qquad \overset{t\to\infty}{\longrightarrow} \qquad e^{-\nu}$$

On the other hand :

$$\lim_{t\to\infty} t^{m^2}p_t^m(x,2y) = \lim_{t\to\infty}\frac{e^{-\mathrm{tr}(x)/2t}}{\Gamma_m(m)}e^{-\mathrm{tr}(y)}{}_0F_1^{(1)}(m,\frac{xy}{2t}) = \frac{e^{-\mathrm{tr}(y)}}{\Gamma_m(m)}$$

Finally :

$$\begin{aligned}
&Q_x^m\left[\exp\left(-\frac{2\nu^2}{(m\log t)^2}\int_0^t \mathrm{tr}(X_s^{-1})ds\right)\right] \\
&= \int_{y>0} Q_x^m\left(\exp(-\frac{2\nu^2}{(m\log t)^2}\int_0^t \mathrm{tr}(X_s^{-1})ds|X_t = y\right)p_t^m(x,y)dy \\
&= \int_{y>0} Q_x^m\left(\exp(-\frac{2\nu^2}{(m\log t)^2}\int_0^t \mathrm{tr}(X_s^{-1})ds|X_t = ty\right)t^{m^2}p_t^m(x,ty)dy \overset{t\to\infty}{\longrightarrow} e^{-\nu},
\end{aligned}$$

by dominated convergence Theorem. ■

5.2. Negative Indices. Take $0 < a \le \det(x)$. Similarly as in paragraph 5.1 with $H_t = -\nu\sqrt{X_t}^{-1}, 0 < \nu < 1$, shows that

$$Q_x^{m-\nu}|_{\mathscr{F}_{t\wedge T_a}} = \left(\frac{\det(x)}{\det(X_{t\wedge T_a})}\right)^{\nu/2}\exp\left(-\frac{\nu^2}{2}\int_0^{t\wedge T_a}\mathrm{tr}(X_s{}^{-1})ds\right)Q_x^m|_{\mathscr{F}_{t\wedge T_a}}$$

where $T_a := \inf\{t, \det(X_t) = a\}$. Letting $a \to 0$ and using the fact that $T_0 = \infty$ a.s under Q_x^m, we get :

$$\begin{aligned} Q_x^{m-\nu}|_{\mathscr{F}_{t\wedge T_0}} &= \left(\frac{\det(x)}{\det(X_t)}\right)^{\nu/2} \exp\left(\frac{\nu^2}{2}\int_0^t \operatorname{tr}(X_s{}^{-1})ds\right) Q_x^m|_{\mathscr{F}_t} \\ &= \left(\frac{\det(x)}{\det(X_t)}\right)^{\nu} Q_x^{m+\nu}|_{\mathscr{F}_t} \end{aligned}$$

PROPOSITION 5.2. *For all $t > 0$ and $0 < \nu < 1$,*

$$Q_x^{m-\nu}(T_0 > t) = \frac{\Gamma_m(m)}{\Gamma_m(m+\nu)} \det(\frac{x}{2t})^{\nu} {}_1F_1(\nu, m+\nu, -\frac{x}{2t})$$

Proof : From the absolute-continuity relation above, we deduce that :

$$Q_x^{m-\nu}(T_0 > t) = Q_x^{m+\nu}\left(\left(\frac{\det(x)}{\det(X_t)}\right)^{\nu}\right),$$

On the other hand, using the expression of the semi-group, one has :

$$\begin{aligned} Q_x^{\delta}(\det(X_t)^s) &= (2t)^{ms}\frac{\Gamma_m(s+\delta)}{\Gamma_m(\delta)} {}_1F_1^{(1)}(-s;\delta;-\frac{x}{2t}) \\ &= (2t)^{ms}\frac{\Gamma_m(s+\delta)}{\Gamma_m(\delta)} \exp(-\operatorname{tr}(\frac{x}{2t})) {}_1F_1^{(1)}(\delta+s;\delta;\frac{x}{2t}) \end{aligned}$$

by Kummer relation (see Th 7. 4. 3 in [**52**]). Taking $s = -\nu$, we are done. ■

6. Generalized Hartman-Watson Law

The generalized Hartman-Watson law is defined as the law of

$$\int_0^t \operatorname{tr}(X_s^{-1})ds \qquad \text{under} \qquad Q_x^m(\cdot|X_t = y).$$

Its Laplace transform is given by :
(23)

$$Q_x^m(\exp\left(\frac{-\nu^2}{2}\int_0^t \operatorname{tr}(X_s^{-1})ds\right)|X_t = y) = \frac{\Gamma_m(m)}{\Gamma_m(m+\nu)} \det(z)^{\nu/2} \frac{{}_0F_1^{(1)}(m+\nu, z)}{{}_0F_1^{(1)}(m, z)}$$

$z = xy/4t^2$. Recall that for $m = 1$, this is the well-known Hartman-Watson law which appears when computing asiatic options as the inverse of the random time change involved in Lamperti representation ([**80**]). Its density was computed by Yor (cf [**115**]). Here, we will investigate the case $m = 2$. First, the Gross and Richards formula writes ([**62**]) :

$${}_0F_1^{(1)}(m+\nu, z) = \frac{\det(z_i^{m-j} {}_0\mathscr{F}_1(m+\nu-j+1, z_i))}{V(z)},$$

where (z_i) denote the eigenvalues of z and $V(z) = \prod_{i<j}(z_i - z_j)$ is the Vandermonde determinant . Noting that $\Gamma_m(m+\nu) = \prod_{j=1}^{m} \Gamma(m+\nu-j+1)$, then :

$$(23) = \frac{\det(z_i^{(m-j)/2} I_{m+\nu-j}(2\sqrt{z_i}))}{\det(z_i^{(m-j)/2} I_{m-j}(2\sqrt{z_i}))}$$

Without loss of generality, we will take $t = 1$.

PROPOSITION 6.1. *For $m = 2$, let $\lambda_1 > \lambda_2$ be the eigenvalues of $\sqrt{xy}$. Then, the density of the generalized Hartman-Watson law is given by :*

$$f(v) = \frac{\sqrt{\lambda_1\lambda_2}v}{p\pi\sqrt{2\pi v^3}} \frac{\int_0^1\int_0^\infty z\sinh(p\sqrt{1-z^2})e^{-2\sqrt{\lambda_1\lambda_2}z\cosh y}e^{-\frac{2(y^2-\pi^2)}{v}}(\sinh y)\sin(\frac{4\pi y}{v})dzdy}{\int_0^1\int_0^1 \frac{u\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}} I_0(2\sqrt{\lambda_1\lambda_2}ux)dudx},$$

for $v > 0$, where $p = \lambda_1 - \lambda_2$. Furthermore, if $\lambda_1 = \lambda_2 := \lambda$, then :

$$f(v) = \frac{4\lambda v e^{\frac{2\pi^2}{v}}}{\pi^2\sqrt{2\pi v^3}} \frac{\int_0^\infty g(y)e^{-\frac{2y^2}{v}}(\sinh y)\sin(\frac{4\pi y}{v})dy}{{}_1\mathscr{F}_2(\frac{1}{2};1;2;\lambda^2)},$$

where

$$g(y) = \frac{1}{3} + \frac{\pi}{2}\frac{I_2(2\lambda\cosh y) + \mathbf{L}_2(2\lambda\cosh y)}{2\lambda\cosh y},$$

*and $\mathbf{L}_2$ is the struve function ([**21**], [**98**]).*

Proof : For $m = 2$, (23) reads :

$$(23) = \frac{\lambda_1 I_{\nu+1}(\lambda_1)I_\nu(\lambda_2) - \lambda_2 I_{\nu+1}(\lambda_2)I_\nu(\lambda_1)}{\lambda_1 I_1(\lambda_1)I_0(\lambda_2) - \lambda_2 I_1(\lambda_2)I_0(\lambda_1)},$$

hence, using the integral representations below (see [**21**], p 46) :

$$x(aI_{\nu+1}(ax)I_\nu(bx) - bI_{\nu+1}(bx)I_\nu(ax)) = (a^2 - b^2)\int_0^x uI_\nu(au)I_\nu(bu)du$$

with $x = 1, a = \lambda_1, b = \lambda_2$, and ([**21**] p 304, [**98**] p 734) :

$$\frac{\pi}{2}I_\nu(\frac{a}{2}(\sqrt{b^2+c^2}+b))I_\nu(\frac{a}{2}(\sqrt{b^2+c^2}-b)) = \int_0^a \frac{\cosh(b\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}}I_{2\nu}(cx)dx$$

$$(a > 0, \Re(\nu) > -1)$$

with $a = 1, b = (\lambda_1 - \lambda_2)u := pu$ et $c = 2\sqrt{\lambda_1\lambda_2}u$, the numerator of (23) is then equal to :

$$\frac{2}{\pi}(\lambda_1^2 - \lambda_2^2)\int_0^1\int_0^1 \frac{u\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}}I_{2\nu}(2\sqrt{\lambda_1\lambda_2}ux)dudx.$$

Taking $\nu = 0$, the denominator is then equal to :

$$\frac{2}{\pi}(\lambda_1^2 - \lambda_2^2)\int_0^1\int_0^1 \frac{u\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}}I_0(2\sqrt{\lambda_1\lambda_2}ux)dudx.$$

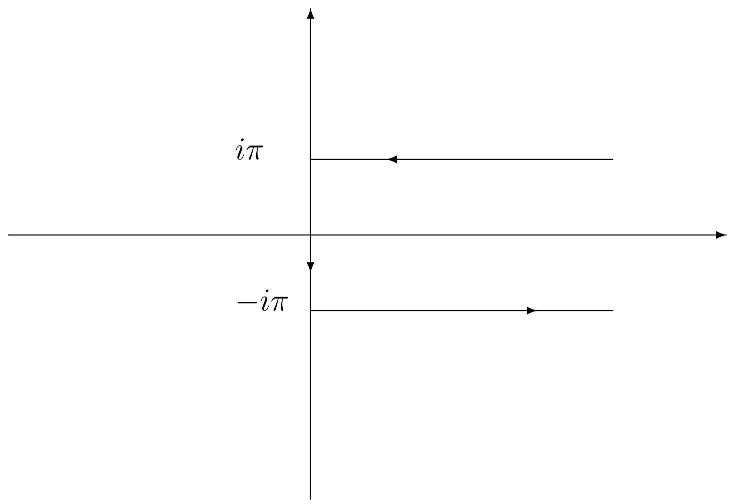
Thus, (23) becomes :

$$\frac{\int_0^1\int_0^1 \frac{u\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}} I_{2\nu}(2\sqrt{\lambda_1\lambda_2}ux)dudx}{\int_0^1\int_0^1 \frac{u\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}} I_0(2\sqrt{\lambda_1\lambda_2}ux)dudx}$$

Now, we only have to use the integral representation of $I_{2\nu}$ (cf [**115**]) :

$$\begin{aligned} I_{2\nu}(2\sqrt{\lambda_1\lambda_2}ux) &= \frac{1}{2i\pi}\int_C e^{2\sqrt{\lambda_1\lambda_2}ux\cosh\omega}e^{-2\nu\omega}d\omega \\ &= \frac{1}{2i\pi}\int_C e^{2\sqrt{\lambda_1\lambda_2}ux\cosh\omega}\int_0^\infty \frac{2\omega e^{-v\nu^2/2}}{(2\pi v^3)^{1/2}}e^{-\frac{2\omega^2}{v}}dvd\omega \end{aligned}$$

where C is the contour indicated in [**115**] :



As a result, the density function is given by :

$$f(v) = \frac{1}{i\pi\sqrt{2\pi v^3}}\frac{\int_0^1\int_0^1\int_C u\omega\frac{\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}}e^{2\sqrt{\lambda_1\lambda_2}ux\cosh\omega}e^{-\frac{2\omega^2}{v}}dudxd\omega}{\int_0^1\int_0^1 \frac{u\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}} I_0(2\sqrt{\lambda_1\lambda_2}ux)dudx}\mathbf{1}_{\{v>0\}}$$

We can simplify this expression by integrating over C (cf [**115**]) :

$$\frac{\sqrt{\lambda_1\lambda_2}v}{\pi\sqrt{2\pi v^3}}\frac{\int_0^1\int_0^1\int_0^\infty u^2x\frac{\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}}e^{-2\sqrt{\lambda_1\lambda_2}ux\cosh y}e^{-\frac{-2(y^2-\pi^2)}{v}}(\sinh y)\sin(\frac{4\pi y}{v})dudxdy}{\int_0^1\int_0^1 \frac{u\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}} I_0(2\sqrt{\lambda_1\lambda_2}ux)dudx}$$

Setting $z = ux$, we obtain

$$\frac{\sqrt{\lambda_1\lambda_2}v}{\pi\sqrt{2\pi v^3}}\frac{\int_0^1\int_0^u\int_0^\infty z\frac{u\cosh(p\sqrt{u^2-z^2})}{\sqrt{u^2-z^2}}e^{-2\sqrt{\lambda_1\lambda_2}z\cosh y}e^{-\frac{2(y^2-\pi^2)}{v}}(\sinh y)\sin(\frac{4\pi y}{v})dudzdy}{\int_0^1\int_0^1\frac{u\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}}I_0(2\sqrt{\lambda_1\lambda_2}ux)dudx},$$

that, an integration with respect to u gives

$$\frac{\sqrt{\lambda_1\lambda_2}v}{p\pi\sqrt{2\pi v^3}}\frac{\int_0^1\int_0^\infty z\sinh(p\sqrt{1-z^2})e^{-2\sqrt{\lambda_1\lambda_2}z\cosh y}e^{-\frac{2(y^2-\pi^2)}{v}}(\sinh y)\sin(\frac{4\pi y}{v})dzdy}{\int_0^1\int_0^1\frac{u\cosh(pu\sqrt{1-x^2})}{\sqrt{1-x^2}}I_0(2\sqrt{\lambda_1\lambda_2}ux)dudx}.$$

Now, we prove the second part. In this case, $p = 0$ and we shall compute :

$$\frac{\lambda ve^{\frac{2\pi^2}{v}}}{\pi\sqrt{2\pi v^3}}\frac{\int_0^\infty\int_0^1\int_0^1\frac{u^2x}{\sqrt{1-x^2}}e^{-2\lambda ux\cosh y}e^{-\frac{2y^2}{v}}(\sinh y)\sin(\frac{4\pi y}{v})dudxdy}{\int_0^1\int_0^1\frac{u}{\sqrt{1-x^2}}I_0(2\lambda ux)dudx}$$

Setting $z = ux$, this reads

$$\frac{\lambda ve^{\frac{2\pi^2}{v}}}{\pi\sqrt{2\pi v^3}}\frac{\int_0^\infty\int_0^1 z\sqrt{1-z^2}e^{-2\lambda z\cosh y}e^{-\frac{2y^2}{v}}(shy)sin(\frac{4\pi y}{v})dzdy}{\int_0^1\int_0^1\frac{u}{\sqrt{1-x^2}}I_0(2\lambda ux)dudx},$$

Integrating with respect to z, we get (cf [**98**] p 369) :

$$\frac{\lambda ve^{\frac{2\pi^2}{v}}}{\pi\sqrt{2\pi v^3}}\frac{\int_0^\infty g(y)e^{-\frac{2y^2}{v}}(shy)sin(\frac{4\pi y}{v})dzdy}{\int_0^1\int_0^1\frac{u}{\sqrt{1-x^2}}I_0(2\lambda ux)dudx},$$

For the denominator, we use the fact that $\frac{d}{dz}(zI_1(z)) = zI_0(z)$, which yields :

$$\int_0^1\int_0^1\frac{uI_0(2\lambda ux)}{\sqrt{1-x^2}}dudx = \int_0^1\frac{I_1(2\lambda x)}{2\lambda x\sqrt{1-x^2}}dx$$

Then, the following formula

(24)

$$\int_0^a x^{\alpha-1}(a^2-x^2)^{\beta-1}I_\nu(cx)dx = 2^{-\nu-1}a^{2\beta+\alpha+\nu-2}c^\nu\frac{\Gamma(\beta)\Gamma((\alpha+\nu)/2)}{\Gamma(\beta+(\alpha+\nu)/2)\Gamma(\nu+1)}$$
$$\,_1\mathscr{F}_2(\frac{\alpha+\nu}{2};\beta+\frac{\alpha+\nu}{2};\nu+1;\frac{a^2c^2}{4})$$

taken with $\alpha = 0, a = 1, \beta = 1/2, c = 2\lambda, \nu = 1$ gives :

$$\int_0^1\frac{I_1(2\lambda x)}{2\lambda x\sqrt{1-x^2}}dx = \frac{\pi}{4}\,_1\mathscr{F}_2(\frac{1}{2};1;2;\lambda^2)$$

We can proceed differently by letting $\lambda_1 = \lambda_2 + h$ then substitute λ_1 in (23) :

$$\frac{((\lambda_2+h)I_{\nu+1}(\lambda_2+h)I_\nu(\lambda_2)-\lambda_2I_{\nu+1}(\lambda_2)I_\nu(\lambda_2+h))/h}{((\lambda_2+h)I_1(\lambda_2+h)I_0(\lambda_2)-\lambda_2I_1(\lambda_2)I_0(\lambda_2+h))/h}.$$

Next, we let $h \to 0$. As usually, we first compute the numerator then take $\nu = 0$. Let :

$$\begin{aligned} A &= \lim_{h\to 0} \frac{(\lambda_2 + h)I_{\nu+1}(\lambda_2 + h) - \lambda_2 I_{\nu+1}(\lambda_2)}{h} \\ B &= \lim_{h\to 0} \frac{I_\nu(\lambda_2 + h) - I_\nu(\lambda_2)}{h} \end{aligned}$$

which are equal respectively to $\frac{d}{dx}(xI_{\nu+1}(x))$ et $\frac{d}{dx}(I_\nu(x))$ taken for $x = \lambda$. (We will write λ instead of λ_2).

Using the differentiation formula $\frac{d}{dx}(x^\nu I_\nu(x)) = x^\nu I_{\nu-1}(x)$(cf [**85**],p 110), we get :

$$\begin{aligned} \frac{d}{dx}(xI_{\nu+1}(x)) &= -\nu I_{\nu+1}(x) + xI_\nu(x) \\ \frac{d}{dx}(I_\nu(x)) &= -\frac{\nu}{x} I_\nu(x) + I_{\nu-1}(x), \end{aligned}$$

thus :

$$\begin{aligned} N &= I_\nu(\lambda)(-\nu I_{\nu+1}(\lambda) + \lambda I_\nu(\lambda)) - \lambda I_{\nu+1}(\lambda)(-\frac{\nu}{\lambda} I_\nu(\lambda) + I_{\nu-1}(\lambda)) \\ &= \lambda({I_\nu}^2(\lambda) - I_{\nu+1}(\lambda)I_{\nu-1}(\lambda)) \end{aligned}$$

so that :

$$A = \frac{{I_\nu}^2(\lambda) - I_{\nu+1}(\lambda)I_{\nu-1}(\lambda)}{{I_0}^2(\lambda) - I_1(\lambda)I_{-1}(\lambda)}$$

On the other hand, the product formula holds (see [**98**],p 757) :

$$I_\mu(z)I_\nu(z) = \frac{2}{\pi}\int_0^{\pi/2} \cos((\mu - \nu)\theta) I_{\mu+\nu}(2z\cos\theta)d\theta, \qquad \Re(\mu + \nu) > -1.$$

Consequently, the numerator is written :

$$\begin{aligned} N &= \frac{2}{\pi}\int_0^{\pi/2} (1 - \cos 2\theta) I_{2\nu}(2\lambda\cos\theta)d\theta \\ &= \frac{4}{\pi}\int_0^{\pi/2} (\sin^2\theta) I_{2\nu}(2\lambda\cos\theta)d\theta \\ &= \frac{4}{\pi}\int_0^1 \sqrt{1 - r^2} I_{2\nu}(2\lambda r)dr, \end{aligned}$$

and similarly, the denominator equals to :

$$D = \frac{4}{\pi}\int_0^1 \sqrt{1 - r^2} I_0(2\lambda r)dr.$$

Eventually, the integral representation of I_ν gives :

$$f(u) = \frac{\lambda u e^{2\pi^2/u}}{\pi\sqrt{2\pi u^3}} \frac{\int_0^\infty e^{-2y^2/u} \sinh(y) \sin\left(\frac{4\pi y}{u}\right) \int_0^1 r\sqrt{1-r^2} e^{-2\lambda r \cosh y} dr\, du}{\int_0^1 \sqrt{1-r^2} I_0(2\lambda r) dr}$$
$$= \frac{\lambda u e^{2\pi^2/u}}{\pi\sqrt{2\pi u^3}} \frac{\int_0^\infty g(y) e^{-2y^2/u} \sinh(y) \sin\left(\frac{4\pi y}{u}\right) du}{\int_0^1 \sqrt{1-r^2} I_0(2\lambda r) dr},$$

We can use (24) to see that the denominator is equal to $(\pi/4)_1\mathscr{F}_2(1/2; 2; 1; \lambda^2)$, which recover the result given in the first approach.

7. The Law of T_0

Recall that for $0 < \nu < 1$,

$$Q_x^{m-\nu}(T_0 > t) = \frac{\Gamma_m(m)}{\Gamma_m(m+\nu)} \det(\frac{x}{2t})^\nu {}_1F_1(\nu, m+\nu, -\frac{x}{2t})$$

Proposition 7.1. *Let $m = 2$ and $\lambda_1 > \lambda_2$ be the eigenvalues of x. The density of $S_0 := 1/(2T_0)$ under $Q_x^{m-\nu}$ is given by :*

$$f(u) = \frac{(\lambda_1\lambda_2)^\nu u^{2\nu-2} e^{-(\lambda_1+\lambda_2)u}}{\Gamma(\nu+1)\Gamma(\nu)} \frac{{}_1\mathscr{F}_1(2, \nu+1, \lambda_1 u) - {}_1\mathscr{F}_1(2, \nu+1, \lambda_2 u)}{(\lambda_1 - \lambda_2)}$$

Corollary 7.1. *If $\lambda_1 = \lambda_2 := \lambda$, f writes :*

$$f(u) = \frac{2\lambda^{2\nu} u^{2\nu-1} e^{-\lambda u}}{\Gamma(\nu+2)\Gamma(\nu)} {}_1\mathscr{F}_1(\nu-1, \nu+2, -\lambda u)$$

Proof : With the help of Gross-Richards Formula, it follows that :

$$Q_x^{m-\nu}(S_0 \le u) = \frac{(\lambda_1\lambda_2)^\nu}{(\lambda_1-\lambda_2)\Gamma_2(\nu+2)} u^{2\nu} (\lambda_1 {}_1\mathscr{F}_1(\nu, \nu+2, -\lambda_1 u) {}_1\mathscr{F}_1(\nu-1, \nu+1, -\lambda_2 u)$$
$$- \lambda_2 {}_1\mathscr{F}_1(\nu, \nu+2, -\lambda_2 u) {}_1\mathscr{F}_1(\nu-1, \nu+1, -\lambda_1 u)),$$

with S_0 defined above. This is a C^∞-function in u. Recall that :

$$\frac{d}{dz} {}_1\mathscr{F}_1(a, b, z) = \frac{a}{b} {}_1\mathscr{F}_1(a+1, b+1, z),$$

thus :

$$f(u) = \frac{d}{du} Q_x^{m-\nu}(S_0 \le u) = K(\nu, \lambda_1, \lambda_2) u^{2\nu-1}(A - B)$$

where

$$
\begin{aligned}
K(\nu,\lambda_1,\lambda_2) &= \frac{(\lambda_1\lambda_2)^\nu}{\Gamma_2(\nu+2)(\lambda_1-\lambda_2)}\\
A &= 2\nu((\lambda_{1\,1}\mathscr{F}_1(\nu,\nu+2,-\lambda_1 u)_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_2 u)\\
&- \lambda_{2\,1}\mathscr{F}_1(\nu,\nu+2,-\lambda_2 u)_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_1 u))\\
B &= \frac{\nu}{\nu+2}((\lambda_1^2 u_1\mathscr{F}_1(\nu+1,\nu+3,-\lambda_1 u)_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_2 u)\\
&- \lambda_2^2 u_1\mathscr{F}_1(\nu+1,\nu+3,-\lambda_2 u)_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_1 u)).
\end{aligned}
$$

Then, we use the contiguous relation :

$$b\,{}_1\mathscr{F}_1(a,b,z) - b\,{}_1\mathscr{F}_1(a-1,b,z) = z\,{}_1\mathscr{F}_1(a,b+1,z)$$

to see that

$$\lambda_1 u_1\mathscr{F}_1(\nu+1,\nu+3,-\lambda_1 u) = (\nu+2)({}_1\mathscr{F}_1(\nu,\nu+2,-\lambda_1 u) - {}_1\mathscr{F}_1(\nu+1,\nu+2,-\lambda_1 u))$$

$$\lambda_2 u_1\mathscr{F}_1(\nu+1,\nu+3,-\lambda_2 u) = (\nu+2)({}_1\mathscr{F}_1(\nu,\nu+2,-\lambda_2 u) - {}_1\mathscr{F}_1(\nu+1,\nu+2,-\lambda_2 u))$$

implies that :

$$f(u) = K_1(\nu,\lambda_1,\lambda_2)u^{2\nu-1}(C+D-E-F)$$

where

$$
\begin{aligned}
K_1(\nu,\lambda_1,\lambda_2) &= \frac{\nu(\lambda_1\lambda_2)^\nu}{\Gamma_2(\nu+2)(\lambda_1-\lambda_2)}\\
C &= \lambda_{1\,1}\mathscr{F}_1(\nu,\nu+2,-\lambda_1 u)_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_2 u)\\
D &= \lambda_{1\,1}\mathscr{F}_1(\nu+1,\nu+2,-\lambda_1 u)_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_2 u)\\
E &= \lambda_{2\,1}\mathscr{F}_1(\nu,\nu+2,-\lambda_2 u)_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_1 u)\\
F &= \lambda_{2\,1}\mathscr{F}_1(\nu+1,\nu+2,-\lambda_2 u)_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_1 u),
\end{aligned}
$$

Applying again the contiguous relation above, one has :

$$\lambda_1 u_1\mathscr{F}_1(\nu+1,\nu+2,-\lambda_1 u) = (\nu+1)({}_1\mathscr{F}_1(\nu,\nu+1,-\lambda_1 u) - {}_1\mathscr{F}_1(\nu+1,\nu+1,-\lambda_1 u))$$

$$\lambda_2 u_1\mathscr{F}_1(\nu+1,\nu+2,-\lambda_2 u) = (\nu+1)({}_1\mathscr{F}_1(\nu,\nu+1,-\lambda_2 u) - {}_1\mathscr{F}_1(\nu+1,\nu+1,-\lambda_2 u))$$

$$\lambda_2 u_1\mathscr{F}_1(\nu,\nu+2,-\lambda_2 u) = (\nu+1)({}_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_2 u) - {}_1\mathscr{F}_1(\nu,\nu+1,-\lambda_2 u))$$

$$\lambda_1 u_1\mathscr{F}_1(\nu,\nu+2,-\lambda_1 u) = (\nu+1)({}_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_1 u) - {}_1\mathscr{F}_1(\nu,\nu+1,-\lambda_1 u))$$

substituting in the expression of f, we obtain :

$$f(u) = K_2(\nu,\lambda_1,\lambda_2)u^{2\nu-2}(G-H),$$

where

$$
\begin{aligned}
K_2(\nu,\lambda_1,\lambda_2) &= \frac{\nu(\nu+1)(\lambda_1\lambda_2)^\nu}{\Gamma_2(\nu+2)(\lambda_1-\lambda_2)}\\
G &= {}_1\mathscr{F}_1(\nu+1,\nu+1,-\lambda_2 u)_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_1 u)\\
H &= {}_1\mathscr{F}_1(\nu+1,\nu+1,-\lambda_1 u)_1\mathscr{F}_1(\nu-1,\nu+1,-\lambda_2 u)
\end{aligned}
$$

The density expression follows from :

$$\begin{aligned}\Gamma_2(\nu+2) &= \Gamma(\nu+2)\Gamma(\nu+1) = \nu(\nu+1)\Gamma(\nu+1)\Gamma(\nu)\\ {}_1\mathscr{F}_1(a,a,z) &= e^{-z}, \qquad {}_1\mathscr{F}_1(a,b,-z) = e^{-z}{}_1\mathscr{F}_1(b-a,b,z),\end{aligned}$$

The case $\lambda_1 = \lambda_2$ is treated in the same way as before (for the Hartman-Watson law). In fact , writing $\lambda_1 = \lambda_2 + h$ and letting $h \to 0$, we see that the density is given by :

$$\begin{aligned}f(u) &= \frac{\lambda^{2\nu}u^{2\nu-2}e^{-2\lambda u}}{\Gamma(\nu+1)\Gamma(\nu)}\frac{d}{d\lambda}\,{}_1\mathscr{F}_1(2,\nu+1,\lambda u) = \frac{2\lambda^{2\nu}u^{2\nu-1}e^{-2\lambda u}}{\Gamma(\nu+2)\Gamma(\nu)}{}_1\mathscr{F}_1(3,\nu+2,\lambda u)\\ &= \frac{2\lambda^{2\nu}u^{2\nu-1}e^{-\lambda u}}{\Gamma(\nu+2)\Gamma(\nu)}{}_1\mathscr{F}_1(\nu-1,\nu+2,-\lambda u) \qquad \blacksquare\end{aligned}$$

Remark. Noting that :

$$Q_x^{m-\nu}(e^{-rS_0}) = r\int_0^\infty e^{-ru}Q_x^{m-\nu}(S_0 \le u)du,$$

we can derive the Laplace transform of S_0 from its distribution function. Integrals of confluent hypergeometric products appear and give rise to the so-called Appell function (or Lauricella function) F_2. In fact, (see [**98**]) :

$$\int_0^\infty u^{2\nu}e^{-(\lambda_1+\lambda_2+r)u}{}_1\mathscr{F}_1(2,\nu+2,\lambda_1 u){}_1\mathscr{F}_1(2,\nu+1,\lambda_2 u)du = K\,F_2(a,b,c,d;x;y)$$

However, using the density function expression in the case $\lambda_1 = \lambda_2$, we get

$$\begin{aligned}Q_x^{m-\nu}(e^{-rS_0}) &= \frac{2\lambda^{2\nu}}{\Gamma(\nu+2)\Gamma(\nu)}\int_0^\infty u^{2\nu-1}e^{-(r+2\lambda)u}{}_1\mathscr{F}_1(3,\nu+2,\lambda u)du\\ &\overset{(1)}{=} \frac{2\lambda^{2\nu}}{\Gamma(\nu+2)\Gamma(\nu)}\sum_{n\ge0}\frac{(3)_n}{(\nu+2)_n}\frac{\lambda^n}{n!}\int_0^\infty u^{2\nu+n-1}e^{-(r+2\lambda)u}du\\ &= \frac{2\lambda^{2\nu}\Gamma(2\nu)}{\Gamma(\nu+2)\Gamma(\nu)(2\lambda+r)^{2\nu}}\sum_{n\ge0}\frac{(3)_n(2\nu)_n}{(\nu+2)_n n!}\left(\frac{\lambda}{2\lambda+r}\right)^n\\ &= \frac{\Gamma(2\nu)}{2^{2\nu-1}\Gamma(\nu+2)\Gamma(\nu)}\left(\frac{2\lambda}{2\lambda+r}\right)^{2\nu}{}_2\mathscr{F}_1(2\nu,3,\nu+2;\frac{\lambda}{2\lambda+r})\\ &\overset{(2)}{=} \frac{2\Gamma(2\nu)}{\Gamma(\nu+2)\Gamma(\nu)}\,{}_2\mathscr{F}_1(2\nu,\nu-1,\nu+2;-\frac{\lambda}{\lambda+r}),\end{aligned}$$

where in (1), we used Fubini-Tonelli theorem and in (2), the Pfaff transformation. (see [**85**])

We close this paper by computing the law, under $Q_x^{m+\nu}$, of

$$A_t := \inf\{u, H_u := \int_0^u \mathrm{tr}(X_s^{-1})ds > t\}$$

taken at an exponential random time with parameter $\mu^2/2$, say T_μ independent of (X_t). Using absolute continuity relations, one gets :

$$\begin{aligned}
Q_x^{m+\nu}(A_{T_\mu} > u) &= Q_x^{m+\nu}\left(\frac{\mu^2}{2}\int_0^\infty e^{-\mu^2 t/2}\mathbf{1}_{\{A_t>u\}}du\right) \\
&= Q_x^{m+\nu}\left(\frac{\mu^2}{2}\int_0^\infty e^{-\mu^2 t/2}\mathbf{1}_{\{t>H_u\}}du\right) \\
&= Q_x^{m+\nu}(e^{-\mu^2 H_u/2}) \\
&= Q_x^m(\left(\frac{\det(X_u)}{\det(x)}\right)^{\nu/2} e^{-\frac{\mu^2+\nu^2}{2}H_u}),
\end{aligned}$$

Setting $\sigma^2 = \mu^2 + \nu^2$, it reads :

$$\begin{aligned}
&Q_x^{m+\nu}(A_{T_\mu} > u) = Q_x^{m+\sigma}(\left(\frac{\det(X_u)}{\det(x)}\right)^{(\nu-\sigma)/2}) \\
&= (\det(x))^{(\sigma-\nu)/2}(2u)^{m(\nu-\sigma)/2}\frac{\Gamma_m((m+\sigma+\nu)/2)}{\Gamma_m(m+\sigma)}{}_1F_1^{(1)}(\frac{\sigma-\nu}{2}, m+\sigma, -\frac{x}{2u}).
\end{aligned}$$

Hence, the distribution function of $Y := 1/(2A_{T_\mu})$ is given by :

$$Q_x^{m+\nu}(Y < t) = (\det(x))^{(\sigma-\nu)/2}t^{m(\sigma-\nu)/2}\frac{\Gamma_m((m+\sigma+\nu)/2)}{\Gamma_m(m+\sigma)}{}_1F_1^{(1)}(\frac{\sigma-\nu}{2}, m+\sigma, -tx),$$

and for $m = 2$,

$$Q_x^{\nu+2}(Y < t) = (\det(x))^{(\sigma-\nu)/2}t^{(\sigma-\nu)}\frac{\Gamma_m((\sigma+\nu+2)/2)}{\Gamma_m(\sigma+2)}{}_1F_1^{(1)}(\frac{\sigma-\nu}{2}, \sigma+2, -tx).$$

Then, applying Gross-Richards formula, we obtain :

$$\begin{aligned}
Q_x^{m+\nu}(Y < t) = &\frac{(x_1x_2)^{(\sigma-\nu)/2)}}{x_1 - x_2}\frac{((\sigma+\nu)/2)\Gamma^2((\sigma+\nu)/2)}{(\sigma+1)\Gamma^2(\sigma+1)}t^{(\sigma-\nu)/2} \\
&[x_{11}\mathscr{F}_1(\frac{\sigma-\nu}{2}, \sigma+2, -tx_1)_1\mathscr{F}_1(\frac{\sigma-\nu}{2}-1, \sigma+1, -tx_2) - \\
&x_{21}\mathscr{F}_1(\frac{\sigma-\nu}{2}, \sigma+2, -tx_2)_1\mathscr{F}_1(\frac{\sigma-\nu}{2}-1, \sigma+1, -tx_1)].
\end{aligned}$$

Where $x_1 > x_2$ denotes the eigenvalues of x. As usually, this is a C^∞ function, thus, we shall compute its derivative to get the density. following the lines of the previous proof, the density is given by :

$$\begin{aligned}
f(t) = &\frac{(x_1x_2)^b ab}{x_1 - x_2}\frac{\Gamma^2(a)}{\Gamma(a+b+1)}t^{2b-2}[{}_1\mathscr{F}_1(b+1, a+b+1, -tx_2)_1\mathscr{F}_1(b-1, a+b+1, -tx_1) \\
&- {}_1\mathscr{F}_1(b+1, a+b+1, -tx_1)_1\mathscr{F}_1(b-1, a+b+1, -tx_2)],
\end{aligned}$$

where $a = \dfrac{\sigma+\nu}{2}$ and $b = \dfrac{\sigma-\nu}{2}$.

Remark. Recall that, in the one-dimensional case, $A_{T_\mu} = R/(2Z)$, where $R \stackrel{\mathscr{L}}{=} \beta(1,a)$ and $Z \stackrel{\mathscr{L}}{==} \gamma_b$, both variables are independent (see [**116**], p. 68).

CHAPITRE 4

Radial Dunkl Processes : Existence and uniqueness, Hitting time, Beta Processes and Random Matrices

We begin with the study of some properties of radial Dunkl process associated to a reduced root system. It is shown that this diffusion is the unique strong solution for all $t \geq 0$ of a SDE with singular drift. Then we study T_0, the first hitting time of the positive chamber : we prove via stochastic calculus an already set result by Chybiryakov on the finiteness of T_0. The second and new part part deals with the law of T_0 for which we compute the tail distribution, as well as some insight via stochastic calculus on how root systems are connected with eigenvalues of selfadjoint Brownian matrices, Wishart and matrix Jacobi processes. This gives rise to the so-called β-processes, referring to the Dyson index, and allow us to recover well known results from matrix theory. Next, we use determinantal representations of some special functions to confirm results by Grabiner on BMs in Weyl chambers. While doing this, we write down the generalized Bessel function for the D-type root system. It is worth noting that the β-Jacobi goes beyond the Dunkl scope since on one hand, it involves a non-reduced root system except in the ultraspherical case. On the other hand, we can associate to it a non-flat positive curvature symmetric space and an affine Weyl group. Nevertheless, our existence and uniqueness result remains valid. Finally, we write down its semi group density.

1. Preliminaries

We begin by pointing out some facts on root systems and radial Dunkl processes. We refer to [**103**] for the Dunkl theory, to both [**25**] and [**67**] for a background on root systems and [**33**], [**55**] for facts on radial Dunkl processes. Let $(V, <,>)$ be a finite real Euclidean space of dimension m. A *reduced* root system R is a finite set of non zero vectors spanning V such that :

1 $R \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in R$.

2 $\sigma_\alpha(R) = R$

where σ_α is the reflection with respect to the hyperplane H_α orthogonal to α :

$$\sigma_\alpha(x) = x - 2\frac{< \alpha, x >}{< \alpha, \alpha >}\alpha, \qquad x \in V$$

A simple system Δ is a basis of V which induces a total ordering in R. A root α is positive if it is a positive linear combination of elements of Δ. The set of positive roots is called a positive subsystem and is denoted by R_+. Note that the choice of Δ is not unique and that R_+ is uniquely determined by Δ. The reflection group W is the one generated by all the reflections σ_α for $\alpha \in R$. Recall that W is finite and the only reflections are of the form σ_α for $\alpha \in R$. Given a root system R with associated positive subsytem R_+, let C be the *positive Weyl chamber* defined by :

$$C := \{x \in V\ <\alpha, x>> 0 \,\forall\, \alpha \in R_+\} = \{x \in V\ <\alpha, x>> 0 \,\forall\, \alpha \in \Delta\}$$

and $\overline{C}$ its closure. One of the most important properties is that the convex cone $\overline{C}$ is a fundamental domain, that is each $\lambda \in V$ is conjugate to one and only one $\mu \in \overline{C}$.

The radial Dunkl process is defined as the $\overline{C}$-valued continuous paths Markov process whose generator is given by :

$$\mathscr{L}u(x) = \frac{1}{2}\Delta u(x) + \sum_{\alpha \in R_+} k(\alpha)\frac{<\alpha, \nabla u(x)>}{<\alpha, x>}$$

with boundary conditions $\nabla u(x) \cdot \alpha = 0$ for all $x \in H_\alpha$, $\alpha \in R_+$, $k(\alpha) \geq 0$ is the multiplicity function (invariant under the action of W), and $u \in C_c^2(C)$. When $k(\alpha) = 1$ for all $\alpha \in R$, we recover the BM constrained to stay in C, studied by Grabiner ([**58**]). The semi-group density of X is given by :

$$p_t^k(x, y) = \frac{1}{c_k t^{\gamma+m/2}} e^{-(|x|^2+|y|^2)/2t} D_k^W(x, y) \prod_{\alpha \in R_+} <\alpha, y>^{2k(\alpha)} \tag{25}$$

for $x, y \in C$, where $\gamma = \sum_{\alpha \in R_+} k(\alpha)$,

$$D_k^W(x, y) := \sum_{w \in W} D_k\left(\frac{x}{\sqrt{t}}, \frac{wy}{\sqrt{t}}\right)$$

where D_k denotes the Dunkl kernel and c_k is given by the Macdonald-Mehta integral ([**103**]). Indeed, as $D_k(0, y) = 1$ ([**103**]), one gets

$$t^{\gamma+m/2}c_k = |W| \int_C e^{-|y|^2/2t} \prod_{\alpha \in R_+} <\alpha, y>^{2k(\alpha)}\, dy = \int_{\mathbb{R}^m} e^{-|y|^2/2t} \prod_{\alpha \in R_+} |<\alpha, y>|^{2k(\alpha)} dy$$

since $\mathbb{R}^m = \cup_{w \in W} w\overline{C}$. $D_k^W(x, y)$ is known as *the generalized Bessel function* (up to the constant $|W|$). This process is obtained by projecting the Dunkl process valued in $\mathbb{R}^m$ (which has right-continuous and left-limits paths, see [**55**]) on $\overline{C}$. The latter was already introduced by Rösler ([**103**],[**104**]) and then studied by Gallardo and Yor ([**55**],[**56**]) and Chybiryakov ([**33**]).

To illustrate all these facts and motivate the reader as well, we will provide some well known examples. We start with the *rank one* case ($m = 1$) for which $R = B_1 = \{\pm 1\}$. Hence $k(\alpha) := k_0 \geq 0$ and X is a *Bessel* process ([**101**]) of *index*

$\nu = k_0 - 1/2$. When $k_0 > 0$, it is the unique strong solution of :

$$dX_t = dB_t + \frac{k_0}{X_t}dt, \quad t \geq 0,\, X_0 = x > 0.$$

where B is a standard BM. Another well known multivariate example is described by the A_{m-1}-type root system defined as :

$$A_{m-1} = \{\pm(e_i - e_j)\, 1 \leq i < j \leq m\},$$

with positive and simple systems given by :

$$R_+ = \{e_i - e_j,\, 1 \leq i < j \leq m\} \quad \Delta = \{e_i - e_{i+1},\, 1 \leq i \leq m\}$$

where $(e_i)_{1\leq i\leq m}$ is the standard basis of $\mathbb{R}^m$. V is the hyperplane of $\mathbb{R}^m$ consisting of vectors that coordinates sum to zero. Without loss of generality, one can take $\mathbb{R}^m$ instead of V so that $C = \{x \in \mathbb{R}^m, x_1 > \cdots > x_m\}$. Besides, there is only one orbit and $k(\alpha) := k_1 \geq 0$. Thus, the corresponding radial Dunkl process satisfies :

$$dX_t^i = d\nu_t^i + k_1 \sum_{j\neq i} \frac{dt}{X_t^i - X_t^j} \qquad 1 \leq i \leq m, \quad t < \tau \tag{26}$$

with $X_0^1 > \cdots > X_0^m$, where $(\nu^i)_i$ are independent Brownian motions and τ is *the first collision time* . For strictly positive k_1, this process was deeply studied by Cépa and Lépingle ([**27**], [**28**], [**29**]) : it behaves as m-interacting particles on the real line with electrostatic repulsions proportional to the inverse of the distance separating them. Moreover, when $k_1 = 1, 1/2$ respectively, this process evolves like eigenvalues process of Hermitian (Dyson model) and symmetric Brownian motions ([**48**], [**58**]). It was shown in [**27**] that this SDE has a unique strong solution for all $t \geq 0$ and $k_1 > 0$. When reading the proof in [**27**], one hopes to extend this result for any root system since materials used there are not typical for the A_{m-1}-type. This was the original motivation of this work. Our first result claims that

$$dX_t = dB_t - \nabla\Phi(X_t)dt, \quad X_0 \in C$$

where $\Phi(x) = -\sum_{\alpha\in R_+} k(\alpha)\ln(<\alpha, x>)$, $k > 0$, has a unique strong solution for all $t \geq 0$. At the same time and independently, Chybiryakov and Schapira provide two other proofs : both authors used well posed martingale problems associated respectively with the $\mathbb{R}^m$-valued Dunkl and the radial Heckman-Opdam processes as well as geometric arguments ([**33**], [**99**]). The curious reader will wonder what happens if $k(\alpha) = 0$ for some α ? The answer is that the same result holds up to *the first hitting time* of ∂C, say T_0 ([**33**] p. 37). Next, we are mainly interested in the tail distribution of T_0. Before proceeding, we reprove via stochastic calculus that $T_0 < \infty$ if $k(\alpha) < 1/2$ for at least one $\alpha \in R_+$ (see [**33**] for the original proof using local martingales). More precisely, for such an α, we prove that almost surely, $<\alpha, X_t> \leq Y_t$ for all $t \geq 0$, where Y is a Bessel process of dimension strictly less than 2. At this level, other proofs exist for the above results. To our knowledge, the contents of the remainder of the paper are new. In [**33**], the author derived absolute-continuity relations which allow us to write the tail distribution of T_0 when starting from $x \in C$. A W-invariant analytic x-dependent integral, which value at 0 is given by a Selberg integral, is involved. As far as we know, though $D_k^W(x, y)$ arises as hypergeometric functions for particular root systems (see the end of [**5**]), forward computations are sophisticated and hard. More precisely, we think that it is possible to use the integral formula given in Corollary 2 in [**70**] with the integration range $0 < X_t^1 < \cdots < X_t^m < 1$, known as the Macdonald's conjecture, then perform limit and sums operations. The matrix cases for which the Jack parameter equals to $= 1, 2$ are more handable with the use of properties of zonal polynomials and Schur functions. However, we think that the approach adopted here is more elegant since on one hand, it disgards the special values of the multiplicity function and on the other hand, does not need long hard formulas. It only relies on some properties picked from Dunkl theory. More precisely, it will be shown that the x-dependent integral is an eigenfunction of some operator which involves the generator $\mathscr{L}$ and the so-called *Euler operator* E_1. For some particular root systems, this eigenfunction is identified with some hypergeometric series. A surprising fact is that the eigenoperator can be expressed in terms of a Schrödinger operator $\mathscr{H}$ and its minimal eigenvalue E_{min} (minimal energy) (see [**103**] page 18) :

$$\mathscr{L} - E_1 := \mathscr{L} - \sum_{i=1}^{m} x_i \partial_i = -e^{|x|^2/4}(\mathscr{H} - E_{min})e^{-|x|^2/4}$$

Moreover, $(X_t)_{t \geq 0}$ specializes for some values of k to eigenvalues processes of self-adjoint matrix processes such as symmetric and Hermitian Brownian motions, Wishart and Laguerre and matrix Jacobi processes. In those cases, computations can be performed using the action of orthogonal and unitary groups. Indeed, Jack polynomials fit zonal polynomials and Schur functions when the Jack parameter equals to $1, 2$ respectively (see [**86**]). The two first ones are identified as A_{m-1}-type radial Dunkl processes while Wishart and Laguerre processes are related to the B_m root system. The latter goes beyond the radial Dunkl setting : the reduced root system C_m in a particular case (ultraspheric) is involved and more generally, the non reduced system BC_m. This connection was deeply investigated in [**8**] while identifying special functions associated with root systems with multivariate hypergeometric series. Among them appear multivariate Gauss hypergeometric series and Jacobi polynomials ([**81**]) and these are eigenfunctions of the β-Jacobi generator. The state space is the so-called *principal Weyl alcove* which is now a bounded convex domain and fundamental for the action of the *affine Weyl group*. Hence, the process evolves like particles in an interval. Then, we extend the strong uniqueness Theorem to the Jacobi context. In the remaining part, we derive some properties : we briefly visit the Brownian motion in the principal Weyl alcove which corresponds to multiplicities all equal to 1. Then, an analogous result on the finiteness of the first hitting time of alcoves walls is obtained using similar computations as those for T_0. Finally, we derive the semi group density and discuss some open questions left in [**43**].

2. Radial Dunkl Process : Existence and Uniqueness of a strong solution

THEOREM 2.1. *Let R be a reduced root system. Let :*

$$\Phi(x) = -\sum_{\alpha\in R_+} k(\alpha)\ln(<\alpha, x>) := \sum_{\alpha\in R_+} k(\alpha)\theta(<\alpha, x>), \qquad x \in C$$

where $k(\alpha) > 0$ for all $\alpha \in R_+$. Then the SDE

$$dX_t = dB_t - \nabla\Phi(X_t)dt, \quad X_0 \in C \tag{27}$$

where X is an adapted continuous process valued in $\overline{C}$ and B is a Brownian motion in $\mathbb{R}^m$, has a unique strong solution.

Proof : From Theorem 2. 2 in [**28**], we deduce that :

$$dX_t = dB_t - \nabla\Phi(X_t)dt + n(X_t)dL_t, \quad X_0 \in C \tag{28}$$

where $n(x)$ is a (unitary) inward normal vector to C at x , L is the boundary process defined by :

$$dL_t = \mathbf{1}_{\{X_t\in\partial C\}}dL_t,$$

has a unique strong solution for all $t \geq 0$. Moreover :

$$\mathbb{E}\left[\int_0^T \mathbf{1}_{\{X_t\in\partial C\}}dt\right] = 0 \tag{29}$$

$$\mathbb{E}\left[\int_0^T |\nabla\Phi(X_t)|dt\right] < \infty \tag{30}$$

for all $T > 0$. Thus, it remains to prove that the boundary process vanishes. To proceed, we need two Lemmas.

REMARK. Both Lemmas below discard the reducedness of R. In fact, this assumption figures in the definition of the Dunkl process and originates from analytic purposes like the commutativity of Dunkl operators ([**46**]).

LEMMA 2.1. *Set $dG_t := n(X_t)dL_t$. Then, $\forall\alpha \in R_+$,*

$$\mathbf{1}_{\{<X_t,\alpha>=0\}} < dG_t, \alpha >= 0$$

Proof : The proof is roughly a generalization of the one in [**27**] for $R = A_{m-1}$. In order to convince the reader, we provide an outline. Using the occupation density formula, we may write $(<\alpha, X>\geq 0)$:

$$\int_0^\infty L_t^a(<\alpha, X>)\theta^{'}(a)da =<\alpha,\alpha> \int_0^t \theta^{'}(<\alpha, X_s>)ds$$

where $L_t^a(<\alpha, X>)$ is the local time of the real continuous semimartingale $<\alpha, X>$. On the other hand, the following inequaliy holds (instead of (2.5) in [**27**]) for all $a \in C$:

$$\begin{aligned}
<\nabla\Phi(x), x-a> &= \sum_{\alpha\in R_+} k(\alpha)\theta^{'}(<\alpha, x>) <\alpha, x-a> \\
&\overset{(1)}{\geq} \sum_{\alpha\in R_+} k(\alpha)[b_\alpha\theta^{'}(<\alpha, x>) - c_\alpha <\alpha, x-a> - d_\alpha] \\
&\geq \min_{\alpha\in R_+}(b_\alpha k(\alpha)) \sum_{\alpha\in R_+} \theta^{'}(<\alpha, x>) - |x-a| \sum_{\alpha\in R_+} k(\alpha)c_\alpha|\alpha| - \sum_{\alpha\in R_+} k(\alpha)d\alpha \\
&:= A \sum_{\alpha\in R_+} \theta^{'}(<\alpha, x>) - B|x-a| - C
\end{aligned}$$

by Cauchy-Schwarz inequality, where in (1), we used eq. (2.1) in [**27**] : let g be a convex C^1-function on an open convex set $D \subset \mathbb{R}^m$, then $\forall a \in D$, there exist $b, c, d > 0$ such that for all $x \in D$:

$$<\nabla g(x), x-a> \geq b|\nabla g(x)| - c|x-a| - d$$

Note also that $A > 0$ since $b_\alpha k(\alpha) > 0$ for all $\alpha \in R_+$. Then, the continuity of X, (30) and the inequality above yield :

$$\int_0^t \theta^{'}(<\alpha, X_s>)ds < \infty$$

which implies that :

$$\int_0^\infty L_t^a(<\alpha, X>)\theta^{'}(a)da < \infty$$

Thus, $L_t^0(<\alpha, X>) = 0$ since the function $a \mapsto \theta^{'}(a)$ is not integrable at 0. The next step consists in using Tanaka formula to compute $dZ_t := <\alpha, X_t> -(<\alpha, X_t>)^+$ for $\alpha \in \Delta$:

$$dZ_t = \mathbf{1}_{\{<\alpha,X_t>=0\}} <\alpha, dB_t> - \mathbf{1}_{\{<\alpha,X_t>=0\}} <\alpha, \nabla\Phi(X_t)> dt + \mathbf{1}_{\{<\alpha,X_t>=0\}} <\alpha, dG_t>$$

It is obvious that the second term vanishes. The first vanishes too since it is a continuous local martingale with null bracket (occupation density formula). As $X_t \in \overline{C}$, then $dZ_t = 0$ a.s. which gives the result. ■

Lemma 2.2. *Let $x \in \partial C$. Then $<n(x), \alpha> \neq 0$ for some $\alpha \in \Delta$ such that $<x, \alpha> = 0$.*

Proof : Let us assume that $<n(x), \alpha> = 0$ for all $\alpha \in \Delta$ such that $<x, \alpha> = 0$. Then, our assumption implies that $<x, \alpha>> 0$ for all $\alpha \in \Delta$ such that $<n(x), \alpha> \neq 0$. If $<n(x), \alpha>< 0$ for these simple roots, then $x - n(x) \in \overline{C}$. By the virtue of the definition of the inward normal $n(x)$ to C at x, i. e,

(31) $$<x-a, n(x)> \leq 0, \quad \forall a \in \overline{C},$$

it follows that $n(x)$ is the null vector which is not possible. Otherwise, choosing :

$$0 < \epsilon < \min_{\alpha/<n(x),\alpha>>0} \frac{< x, \alpha >}{< n(x), \alpha >}$$

we claim that $a := x - \epsilon n(x) \in \partial C$. Arguing as before, we are done. ∎

Now we proceed to end the proof of the Theorem. Note first that $\partial C = \cup_{\alpha\in\Delta} H_\alpha$ so that :

$$\mathbf{1}_{\{X_t \in \partial C\}} dL_t \leq \sum_{\alpha\in\Delta} \mathbf{1}_{\{<X_t,\alpha>=0\}} dL_t.$$

If $X_t \in H_\alpha$ for one and only one $\alpha \in \Delta$. Then, $n(X_t) = \alpha/||\alpha||$ and Lemma 2.1 gives

$$\mathbf{1}_{\{<X_t,\alpha>=0\}} < dG_t, \alpha >= \mathbf{1}_{\{<X_t,\alpha>=0\}} ||\alpha|| dL_t = 0$$

Hence, the boundary process vanishes. More generally, we can use the inequality above and write

$$\begin{aligned} 0 \leq L_t &\leq \sum_{\alpha\in\Delta} \int_0^t \mathbf{1}_{\{<X_s,\alpha>=0\}} \mathbf{1}_{\{<n(X_s),\alpha>\neq 0\}} dL_s \\ &= \sum_{\alpha\in\Delta} \int_0^t \mathbf{1}_{\{<n(X_s),\alpha>\neq 0\}} \frac{1}{< n(X_s), \alpha >} \mathbf{1}_{\{<X_s,\alpha>=0\}} < dG_s, \alpha >= 0 \end{aligned}$$

by Lemma 2.1. ∎

Remark. When $m = 1$, $(X_t)_{t\geq 0}$ is a Bessel process of dimension $\delta = 2k_0 + 1$ and $k_0 > 0 \Leftrightarrow \delta > 1$. It is well known that the local time vanishes (see Ch. XI in [**101**]) which fits our result.

3. Finiteness of the first hitting time of the Weyl chamber

Let $T_0 := \inf\{t > 0, X_t \in \partial C\}$ be the first hitting time of the Weyl chamber. It was shown in [**33**] (see p. 30) that $T_0 = \infty$ almost surely if $k(\alpha) \geq 1/2$ for all $\alpha \in R_+$. In [**28**], where $R = A_{m-1}$ and $T_0 = \inf\{t > 0,\ X_t^i = X_t^j \text{ for some } (i,j)\}$, authors showed that $T_0 < \infty$ a.s. if and only if $0 < k_1 < 1/2$. More generally, the following holds (see [**33**] p. 75 for the original proof) :

Proposition 3.1. *Let $\alpha_0 \in \Delta$ and $T_{\alpha_0} := \inf\{t > 0, < \alpha_0, X_t >= 0\}$ such that $T_0 = \inf_{\alpha_0\in\Delta} T_{\alpha_0}$. If $0 < k(\alpha_0) < 1/2$, then $(< \alpha_0, X_t >)_{t\geq 0}$ hits almost surely 0. In particular, $T_0 < T_{\alpha_0} < \infty$ a. s.*

Proof : assume $k(\alpha) > 0$ for all $\alpha \in R$ and let $\alpha_0 \in \Delta$. Our scheme is roughly the same as that used in [**28**], thus we shall show that the process $< \alpha_0, X >$ is almost surely less than or equal to a Bessel process with dimension $2k(\alpha_0) + 1$. The result follows from the fact that $2k(\alpha_0) + 1 < 2$ when $k(\alpha) < 1/2$. For this, we use the SDE (27). For all $t \geq 0$,

$$d < \alpha_0, X_t > = ||\alpha_0|| d\gamma_t + \sum_{\alpha \in R_+} k(\alpha) \frac{< \alpha, \alpha_0 >}{< \alpha, X_t >} dt$$
$$= ||\alpha_0|| d\gamma_t + k_0 \frac{||\alpha_0||^2}{< \alpha_0, X_t >} dt + \sum_{\alpha \in R_+ \setminus \alpha_0} k(\alpha) \frac{< \alpha, \alpha_0 >}{< \alpha, X_t >} dt$$

where k_0 is the value of $k(\alpha_0)$ corresponding to the conjugacy class of α_0. Set

$$R = \cup_{j=1}^p R^j$$

where R^j, $1 \leq j \leq p$ denote the conjugacy classes of R under the W-action, then

$$R_+ = \cup_{i=1}^p R_+^j$$

so that :

$$d < \alpha_0, X_t >= ||\alpha_0|| d\gamma_t + k_0 \frac{||\alpha_0||^2}{< \alpha_0, X_t >} dt + \sum_{j=1}^p k_j \sum_{\alpha \in R_+^j \setminus \alpha_0} \frac{< \alpha, \alpha_0 >}{< \alpha, X_t >} dt$$

For a conjugacy class R^j and $\alpha \in R^j$, if $< \alpha, \alpha_0 >= a(\alpha) > 0$ then, it is easy to check that $< \sigma_0(\alpha), \alpha_0 >= -a(\alpha)$ where σ_0 is the reflection with respect to the orthogonal hyperplane H_{α_0} defined by :

$$\sigma_0(x) = x - 2 \frac{< x, \alpha_0 >}{< \alpha_0, \alpha_0 >} \alpha_0$$

Note that $\sigma_0(\alpha)$ belongs to the same conjugacy class of α and that $\sigma_0(\alpha) \in R_+$ for $\alpha \in R_+ \setminus \alpha_0$. Indeed, $\sigma_0(R_+ \setminus \alpha_0) = R_+ \setminus \alpha_0$ for all $\alpha_0 \in \Delta$ (see Proposition 1. 4 in [**67**]). Hence,

$$d < \alpha_0, X_t >= ||\alpha_0|| d\gamma_t + k_0 \frac{||\alpha_0||^2}{< \alpha_0, X_t >} dt - \sum_{j=1}^p k_j \sum_{\substack{\alpha \in R_+^j \setminus \alpha_0 \\ a(\alpha) > 0}} \frac{a(\alpha) < \alpha - \sigma_0(\alpha), X_t >}{< \alpha, X_t >< \sigma_0(\alpha), X_t >} dt$$

Furthermore,

$$\alpha - \sigma_0(\alpha) = 2 \frac{< \alpha, \alpha_0 >}{< \alpha_0, \alpha_0 >} \alpha_0 \quad \Rightarrow \quad < \alpha - \sigma_0(\alpha), X_t >= 2a(\alpha) \frac{< \alpha_0, X_t >}{||\alpha_0||^2}$$

Consequently, one gets :

$$d < \alpha_0, X_t >= ||\alpha_0|| d\gamma_t + k_0 \frac{||\alpha_0||^2}{< \alpha_0, X_t >} dt + F_t \, dt$$

where $F_t < 0$ on $\{T_{\alpha_0} = \infty\}$. Using the comparison Theorem in [**71**] (Proposition 2. 18. p. 293 and Exercice 2. 19. p. 294), one claims that $< \alpha_0, X_t > \leq Y^x_{||\alpha_0||^2 t}$ for all $t \geq 0$ on $\{T_{\alpha_0} = \infty\}$, where Y^x is a Bessel process defined on the same probability space with respect to the same Brownian motion, of dimension $2k_0 + 1$

and starting at $Y_0 = x \geq <\alpha_0, X_0>> 0$. This is not possible since a Bessel process of dimension < 2 hits 0 a. s. ([**101**] Chap. XI) ■

Remark. If we remove the assumption $k(\alpha) > 0$ for all $\alpha \in R$, then the SDE (27) can be solved up to time T_0 when starting from $x \in C$ (see [**33**]).

4. The law of T_0

Here, we focus on the tail distribution of T_0 deduced from absolute continuity relations derived in ([**33**]). Recall that (see [**33**]) the *index* of X is defined by $l(\alpha) := k(\alpha) - 1/2$. The last result asserts that if $-1/2 < l(\alpha) < 0$ for some $\alpha \in \Delta$, then $T_0 < \infty$ almost surely. Besides, if $l(\alpha) \geq 0$ for all $\alpha \in \Delta$ then $T_0 = \infty$ almost surely. Taking into account these statements, two major parts are considered : $l(\alpha) \geq 0$ for all $\alpha \in R$ so that the process with index $-l$ hits 0 almost surely, and $l(\alpha) < 0$ for at least one α. The tail distribution involves a W-invariant x-dependent integral. Our line of thinking relies on showing that it is an eigenfunction of an appropriate differential operator. Then, using uniqueness results for some differential equations, the tail distribution is written in A_{m-1} and B_m cases by means of multivariate hypergeometric functions. In the last case, we recover known results from matrix theory for Wishart and Laguerre processes. However, we find it better to postpone this in the next section where links with eigenvalues of matrix-valued processes are detailed.

4.1. A first formula. Let us denote by P_x^l the law of $(X_t)_{t\geq 0}$ starting from $x \in C$. Let E_x^l be the corresponding expectation. Recall that ([**33**], Proposition 2.15.c), if $l(\alpha) \geq 0 \, \forall \alpha \in R_+$, then :

$$\begin{aligned}
P_x^{-l}(T_0 > t) &= E_x^l\left[\left(\prod_{\alpha\in R_+}\frac{<\alpha, X_t>}{<\alpha, x>}\right)^{-2l(\alpha)}\right] \\
&= \prod_{\alpha\in R_+} <\alpha, x>^{2l(\alpha)} \frac{e^{-|x|^2/2t}}{c_k t^{\gamma+m/2}} \int_C e^{-|y|^2/2t} D_k^W\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) \prod_{\alpha\in R_+} <\alpha, y> dy \\
&= \prod_{\alpha\in R_+} <\alpha, x>^{2l(\alpha)} \frac{e^{-|x|^2/2t}}{c_k t^{\gamma'}} \int_C e^{-|y|^2/2} D_k^W\left(\frac{x}{\sqrt{t}}, y\right) \prod_{\alpha\in R_+} <\alpha, y> dy \\
&:= \prod_{\alpha\in R_+} <\alpha, x>^{2l(\alpha)} \frac{e^{-|x|^2/2t}}{c_k t^{\gamma'}} g\left(\frac{x}{\sqrt{t}}\right)
\end{aligned}$$

where $\gamma = \sum_{\alpha\in R_+} k(\alpha)$ and $\gamma' = \gamma - |R_+|/2$.

Though D_k^W is given by hypergeometric functions in the special cases A_{m-1} and B_m (see the end of [**5**]), the *Jack polynomials* defining them prevent us from making computations. However, this may be possible when these polynomials fit, for some values of k, *Zonal polynomials* and *Schur functions* (see [**86**] for definitions). Our main result does not make these restrictions and uses some properties of the Dunkl kernel D_k :

THEOREM 4.1. *Let T_i be the i-th difference Dunkl operator and $\Delta_k = \sum_{i=1}^m T_i^2$ the Dunkl Laplacian ([**103**]). Define :*

$$\mathscr{J}_k^x := -\Delta_k^x + \sum_{i=1}^m x_i \partial_i^x := -\Delta_k^x + E_1^x$$

where $E_1^x := \sum_{i=1}^m x_i \partial_i^x$ is the Euler operator and the superscript indicates the derivative action. Then

$$\mathscr{J}_k^x \left[e^{-|y|^2/2} D_k^W(x,y) \right] = E_1^y \left[e^{-|y|^2/2} D_k^W(x,y) \right]$$

Proof : Recall that if f is W-invariant then $T_i^x f = \partial_i^x f$ and that $T_i^x D_k(x,y) = y_i D_k(x,y)$ (see [**103**]). Then, on one hand :

$$\begin{aligned}
\Delta_k^x D_k^W(x,y) &= \sum_{w \in W} \sum_{i=1}^m (wy)_i T_i^x D_k(x,wy) = \sum_{w \in W} \sum_{i=1}^m (wy)_i^2 D_k(x,wy) \\
&= \sum_{i=1}^m y_i^2 \sum_{w \in W} D_k(x,wy) := p_2(y) D_k^W(x,y)
\end{aligned}$$

On the other hand :

$$\begin{aligned}
E_1^x D_k^W(x,y) &= \sum_{w \in W} \sum_{i=1}^m x_i T_i^x D_k(x,wy) = \sum_{w \in W} \sum_{i=1}^m (x_i)(wy)_i D_k(x,wy) \\
&= \sum_{w \in W} < x, wy > D_k(x,wy) = \sum_{w \in W} < w^{-1}x, y > D_k(x,wy) \\
&= E_1^y D_k^W(x,y)
\end{aligned}$$

where the last equality follows from $D_k(x,wy) = D_k(w^{-1}x,y)$ since $D_k(wx,wy) = D_k(x,y)$ for all $w \in W$. The result follows from an easy computation.

COROLLARY 4.1. *g is an eigenfunction of $-\mathscr{J}_k$ corresponding to the eigenvalue $m + |R_+|$.*

Proof : Theorem 1 and an integration by parts give :

$$\begin{aligned}
-\mathscr{J}_k^x g(x) &= -\int_C E_1^y\left[e^{-|y|^2/2}D_k^W(x,y)\right]\prod_{\alpha\in R_+} <\alpha, y> dy \\
&= -\sum_{i=1}^m \int_C y_i \prod_{\alpha\in R_+} <\alpha, y> \partial_i^y\left[e^{-|y|^2/2}D_k^W(x,y)\right] dy \\
&= \sum_{i=1}^m \int_C e^{-|y|^2/2}D_k^W(x,y)\partial_i\left[y_i \prod_{\alpha\in R_+} <\alpha, y>\right] dy \\
&= \int_C e^{-|y|^2/2}D_k^W(x,y)\prod_{\alpha\in R_+} <\alpha, y> \sum_{i=1}^m\left[1+\sum_{\alpha\in R_+}\frac{\alpha_i y_i}{<\alpha,y>}\right] dy
\end{aligned}$$

and the proof ends by summing over i. ■

– The A_{m-1} case : as mentioned in the introduction, the A_{m-1}-type root system is characterized by :

$$\begin{aligned}
R &= \{\pm(e_i - e_j),\, 1\le i<j\le m\} \qquad R_+ = \{e_i - e_j,\, 1\le i<j\le m\} \\
\Delta &= \{e_i - e_{i+1},\, 1\le i\le m\} \qquad C = \{y\in\mathbb{R}^m,\, y_1 > y_2 > \cdots > y_m\}
\end{aligned}$$

$W = S_m$ is the permutations group and there is one conjugacy class so that $k = k_1 > 0 \Rightarrow \gamma = k_1 m(m-1)/2$. Moreover, the generalized Bessel function [1] is given by ([**5**] p. 212-214, [**30**]) :

$$\frac{1}{|W|}D_k^W(x,y) = {}_0F_0^{(1/k_1)}(x,y) := \sum_{p=0}^{\infty}\sum_{\tau}\frac{J_\tau^{(1/k_1)}(x)J_\tau^{(1/k_1)}(y)}{J_\tau^{(1/k_1)}(1)p!}$$

where $\tau = (\tau_1,\dots,\tau_m)$ is a partition of weight $|\tau| = p$ and length m, $J_\tau^{(1/k_1)}$ is the Jack polynomial of Jack parameter $1/k_1$[2], (see [**5**], [**86**]). Hence, letting V to be the Vandermonde function :

$$P_x^{-l}(T_0 > t) = V(x)^{2k_0-1}\frac{|W|e^{-|x|^2/2t}}{c_k t^{k_0 m(m-1)/2}}\int_C e^{-|y|^2/2}{}_0F_0^{(1/k_0)}(\frac{x}{\sqrt{t}}, y)V(y)dy$$

Besides, $\mathscr{J}_k$ writes on W-invariant functions

$$-\mathscr{J}_k^x = D_0^x - E_1^x := \sum_{i=1}^m \partial_i^{2,x} + 2k_1\sum_{i\neq j}\frac{1}{x_i - x_j}\partial_i - \sum_{i=1}^m x_i\partial_i^x$$

[1]Authors use the change of variable $x \mapsto \sqrt{2}x$, $y \mapsto \sqrt{2}y$ to fit the hypergeometric function obtained when deriving the generating function for Hermite polynomials. This in turn will modify the eigenoperator by a multiplying constant (see p. 183).

[2]With the same notations in [**5**], $k_1 = 2/\alpha$. This can be seen either from the eigenoperator below or from the orthogonality weight function involved in the semi group density.

Finally, since g is W-invariant, then

$$\begin{aligned}(D_0^x - E_1^x)\, g(x) &= m\frac{m+1}{2}g(x),\\ g(0) = \int_C e^{-|y|^2/2}V(y)dy &= \frac{1}{m!}\int_{\mathbb{R}^m} e^{-|y|^2/2}|V(y)|dy\end{aligned}$$

Let us recall that the Gauss hypergeometric function

$${}_2F_1^{(1/k_1)}(e,b,c,z) = \sum_{p=0}^{\infty}\sum_{\tau}\frac{(e)_\tau (b)_\tau}{(c)_\tau}\frac{J_\tau^{(1/k_1)}(z)}{p!}$$

is the unique symmetric eigenfunction that equals to 1 at 0 of (see [**8**] p. 585)
(32)

$$\sum_{i=1}^m z_i(1-z_i)\partial_i^{2,z}+2k_1\sum_{i\neq j}\frac{z_i(1-z_i)}{z_i-z_j}\partial_i^z+\sum_{i=1}^m\left[c-k_1(m-1)-(e+b+1-k_1(m-1))\,z_i\right]\partial_i^z$$

associated to the eigenvalue meb. Letting $z=(1/2)(1-x/\sqrt{b})$, $e=(m+1)/2$ and

$$c = k_1(m-1)+\frac{1}{2}[e+b+1-k_1(m-1)] = \frac{b}{2}+\frac{k_1}{2}(m-1)+\frac{m+3}{4}$$

the resulting function is an eigenfunction of

$$\sum_{i=1}^m(1-\frac{x_i^2}{b})\partial_i^{2,x}+2k_1\sum_{i\neq j}\frac{(1-x_i^2/b)}{x_i-x_j}\partial_i^x-\sum_{i=1}^m(b+\frac{m+3}{2}-k_1(m-1))\frac{x_i}{b}\partial_i^x$$

and $D_0^x - E_1^x$ is the limiting operator as b tends to infinity. Hence,

PROPOSITION 4.1. *For $k_1 \geq 1/2$,*

$$g(x) = g(0)C(m,k_1)\lim_{b\to\infty}{}_2F_1^{(1/k_1)}\left[m+1,b,\frac{b}{2}+\frac{k_1}{2}(m-1)+\frac{m+3}{2},\frac{1}{2}\left(1-\frac{x}{\sqrt{b}}\right)\right]$$

where

$$C(m,k_1)^{-1} = \lim_{b\to\infty}{}_2F_1^{(1/k_1)}\left(m+1,b,\frac{b}{2}+\frac{k_1}{2}(m-1)+\frac{m+3}{2},\frac{1}{2}\right)$$

– The B_m case : This root system is defined by

$$\begin{aligned}R &= \{\pm e_i, \pm e_i \pm e_j,\, 1\leq i<j\leq m\} &\quad R_+ &= \{e_i, 1\leq i\leq m,\, e_i\pm e_j,\, 1\leq i<j\leq m\}\\ \Delta &= \{e_i - e_{i+1},\, 1\leq i\leq m,\, e_m\} &\quad C &= \{y\in\mathbb{R}^m,\, y_1>y_2>\cdots>y_m>0\}\end{aligned}$$

The Weyl group is generated by transpositions and " change sign" reflections $(x_i \mapsto -x_i)$ and there are two conjugacy classes so that $k=(k_0,k_1)\Rightarrow \gamma =$

$mk_0 + m(m-1)k_1$. The generalized Bessel function[3] is given by ([**5**] p. 214) :

$$\frac{1}{|W|}D_k^W(x,y) = {}_0F_1^{(1/k_1)}(k_0 + (m-1)k_1 + \frac{1}{2}, \frac{x^2}{2t}, \frac{y^2}{2t})$$

where

$${}_0F_1^{(1/k_1)}(c,x,y) = \sum_{p=0}^{\infty}\sum_{\tau}(c)_\tau \frac{J_\tau^{(k_1)}(x)J_\tau^{(1/k_1)}(y)}{J_\tau^{(1/k_1)}(1)p!}$$

and $(c)_\tau := \prod_{i=1}^m (c - k_1(i-1))_{\tau_i}$ is the generalized Pochammer symbol (see [**5**]). Then, one has :

$$g(x) = |W|\int_C e^{-|y|^2/2}{}_0F_1^{(1/k_1)}(k_0 + (m-1)k_1 + \frac{1}{2}, \frac{x^2}{2}, \frac{y^2}{2})\prod_{i=1}^m (y_i)V(y^2)dy$$

The eigenoperator writes on W-invariant functions :

$$\begin{aligned} -\mathscr{J}_k^x &= \sum_{i=1}^m \partial_i^{2,x} + 2k_0\sum_{i=1}^m \frac{1}{x_i}\partial_i^x + 2k_1\sum_{i\neq j}\left[\frac{1}{x_i - x_j} + \frac{1}{x_i + x_j}\right]\partial_i^x - E_1^x \\ -\mathscr{J}_k^x g(x) &= m(m+1)g(x), \quad g(0) = \frac{1}{2^m m!}\int_{\mathbb{R}^m} e^{-|y|^2}\prod_{i=1}^m |y_i||V(y^2)|dy. \end{aligned}$$

A change of variable $x_i = \sqrt{2y_i}$ shows that $u(y) := g(\sqrt{2y})$ satisfies

$$\begin{aligned} -\tilde{\mathscr{J}}_k^y u(y) &= m\frac{(m+1)}{2}u(y), \quad g(0) = u(0) \\ -\tilde{\mathscr{J}}_k^y &= \sum_{i=1}^m y_i\partial_i^{2,y} + 2k_1\sum_{i\neq j}\frac{y_i}{y_i - y_j}\partial_i^y + \left(k_0 + \frac{1}{2}\right)\sum_{i=1}^m \partial_i^y - E_1^y. \end{aligned}$$

which implies that :

$$u(y) = u(0){}_1F_1^{(1/k_1)}(\frac{m+1}{2}, k_0 + (m-1)k_1 + \frac{1}{2}, y)$$

where

$${}_1F_1^{(1/k_1)}(b,c,z) = \sum_{p=0}^{\infty}\sum_{\tau}\frac{(b)_\tau}{(c)_\tau}\frac{J_\tau^{(1/k_1)}(z)}{p!}$$

This can be seen from the differential equation (32) and using ([**5**]) :

$$\lim_{e\to\infty}{}_2F_1^{(1/k_1)}(e,b,c,\frac{z}{e}) = {}_1F_1^{(1/k_1)}(b,c,z)$$

[3]there is an erroneous sign in one of the arguments in [**5**]. Moreover, to recover this expression in the B_m case from that given in [**5**], one should make substitutions $a = k_0 - 1/2$, $k_1 = 1/\alpha$, $q = 1 + (m-1)k_1$. We point to the reader that this is different from the one used in [**30**] p. 121.

Finally

$$g\left(\frac{x}{\sqrt{t}}\right) = g(0)\,{}_1F_1^{(1/k_1)}(\frac{m+1}{2}, k_0 + (m-1)k_1 + \frac{1}{2}, \frac{x^2}{2t})$$

Hence, the tail distribution is given by :

Proposition 4.2. *For $k_0, k_1 \geq 1/2$,*

$$P_x^{-l}(T_0 > t) = C_k \prod_{i=1}^{m}\left(\frac{x_i^2}{2t}\right)^{k_0-1/2}\left(V\left(\frac{x^2}{2t}\right)\right)^{2k_1-1} e^{-|x|^2/2t}\,{}_1F_1^{(1/k_1)}(\frac{m+1}{2}, k_0+(m-1)k_1+\frac{1}{2}, \frac{x^2}{2t})$$

Remark. 1/Adopting the notations used in [**5**], one has :

$$-\tilde{\mathscr{J}}_k^y \;=\; D_1^y + (a+1)E_0^y - E_1^y \quad (R = B_m,\, y = x^2),$$

Besides, Theorem 4.1 was derived there differently for both A_{m-1} and B_m cases when proving a generating function Theorem for generalized Hermite and Laguerre polynomials (page 183 and 192, see also [**30**]).

4.2. A second formula. In [**33**] (see Proposition 2.15.b), the author derived another absolute-continuity relation from which we deduce that if $l(\alpha) < 0$ for at least one $\alpha \in R_+$, then

$$\begin{aligned} P_x^l(T_0 > t) &= E_x^0\left[\prod_{\alpha\in R_+}\left(\frac{<\alpha, X_t>}{<\alpha, x>}\right)^{l(\alpha)} \exp\left(-\frac{1}{2}\sum_{\alpha,\gamma\in R_+}\int_0^t \frac{<\alpha,\gamma> l(\alpha)l(\gamma)}{<\alpha, X_s><\gamma, X_s>}ds\right)\right] \\ &= E_x^r\left[\prod_{\alpha\in R_+}\left(\frac{<\alpha, X_t>}{<\alpha, x>}\right)^{l(\alpha)-r(\alpha)} \exp\left(-\frac{1}{2}\sum_{\alpha,\gamma\in R_+}\int_0^t \frac{<\alpha,\gamma> l(\alpha,\gamma)}{<\alpha, X_s><\gamma, X_s>}ds\right)\right] \end{aligned}$$

where the last equality follows from part (c) of the same Proposition, $l(\alpha,\gamma) = l(\alpha)l(\gamma) - r(\alpha)r(\gamma)$ and

$$r(\alpha) = \begin{cases} l(\alpha) & \text{if} \quad l(\alpha) \geq 0 \\ -l(\alpha) & \text{if} \quad l(\alpha) < 0 \end{cases}$$

Then $l(\alpha,\gamma) = 0$ if $l(\alpha)l(\gamma) \geq 0$ and $l(\alpha,\gamma) = -2r(\alpha)r(\gamma)$ else. As a result,

$$P_x^l(T_0 > t) = E_x^r\left[\prod_{\substack{\alpha\in R_+\\ l(\alpha)<0}}\left(\frac{<\alpha, x>}{<\alpha, X_t>}\right)^{2r(\alpha)} \exp\left(\sum_{\substack{\alpha,\gamma\in R_+\\ l(\alpha)l(\gamma)<0}}\int_0^t \frac{<\alpha,\gamma> r(\alpha)r(\gamma)}{<\alpha, X_s><\gamma, X_s>}ds\right)\right]$$

Next, note that the exponential functional equals 1 for both root systems A_{m-1} and B_m. For the first, it is obvious since R consists of one orbit so that $\{\alpha,\, \gamma \in R_+, l(\alpha)l(\gamma) < 0\}$ is empty. This gives the same expression already considered in the previous subsection. For the second, writing $R_+ = \{e_i, 1 \leq i \leq m\} \cup \{e_j \pm e_k, 1 \leq j < k \leq m\}$ so that $< e_i, e_j \pm e_k >= \delta_{ij} \pm \delta_{ik}$ gives

$$S = \sum_{i=1}^{m} \sum_{i<k} \frac{1}{X_t^i} \left[\frac{1}{X_t^i - X_t^k} + \frac{1}{X_t^i + X_t^k} \right] + \sum_{i=1}^{m} \sum_{k<i} \frac{1}{X_t^i} \left[\frac{-1}{X_t^k - X_t^i} + \frac{1}{X_t^k + X_t^i} \right]$$
$$= \sum_{i=1}^{m} \sum_{i<k} \frac{2}{(X_t^i)^2 - (X_t^k)^2} - \sum_{i=1}^{m} \sum_{k<i} \frac{2}{(X_t^k)^2 - (X_t^i)^2} = 0$$

where S stands for the sum between brackets. The reader can also check that this holds for C_m and D_m root systems (see the end of the paper for definitions). However, we restrict ourselves to the B_m -case since, for particular values of the multiplicity function, we will recover a known result from matrix theory (see next section). Let us investigate the case $k_0 < 1/2$, $k_1 \geq 1/2$ for which $l_0 < 0$, $l_1 \geq 0$. One writes :

$$g(x) = \int_C e^{-|y|^2/2} D_k^W(x,y) \prod_{i=1}^{m} (y_i) V^{2k_1}(y^2) dy$$

The machinery used before still applies and gives :

$$-\mathscr{J}_k g = 2m[1 + k_1(m-1)]g$$

Thus

Proposition 4.3. *In the B_m case and for $k_0 < 1/2$, $k_1 \geq 1/2$, one has :*

$$P_x^l(T_0 > t) = C_k \prod_{i=1}^{m} \left(\frac{x_i^2}{2t} \right)^{k_0 - 1/2} e^{-|x|^2/2t} {}_1F_1^{(1/k_1)}(1 + k_1(m-1), k_0 + (m-1)k_1 + \frac{1}{2}, \frac{x^2}{2t})$$

In the remaining case $k_0 \geq 1/2$, $k_1 < 1/2$, the tail distribution writes :

$$g(x) = \int_C e^{-|y|^2/2} D_k^W(x,y) \prod_{i=1}^{m} (y_i)^{2k_0} V(y^2) dy$$

Thus

$$-\mathscr{J}_k g = m[2k_0 + m]g$$

so that

Proposition 4.4. *For $k_0 \geq 1/2, k_1 < 1/2$,*

$$P_x^l(T_0 > t) = C_k V \left(\frac{x^2}{2t} \right)^{2k_1 - 1} e^{-|x|^2/2t} {}_1F_1^{(1/k_1)}(k_0 + \frac{m}{2}, k_0 + (m-1)k_1 + \frac{1}{2}, \frac{x^2}{2t})$$

5. β-processes and random matrices

In the sequel, we will see how eigenvalues of some classical matrix-valued processes and radial Dunkl processes are interelated using SDE. This connection was already checked by physicists throughout eigenvalues probability densities and Fokker-Planck equations for parameter-dependent random matrices ([**30**]). As we mentioned in the introduction, the A_{m-1}-type is connected to symmetric and Hermitian Brownain motions. Set $k := \beta/2$, $\beta > 0$, then such a process will be called β-Dyson, referring to the Dyson model when $\beta = 2$. This parameter is called the Dyson index. Henceforth, we will adopt new notation for the eigenvalues process, we wil write λ instead of X.

5.1. The B_m-type : β-Laguerre processes. The B_m system turns out to be related to eigenvalues of Wishart and Laguerre processes which satisfy the following stochastic differential system (see [**19**],[**37**]) :

$$d\lambda_i(t) = 2\sqrt{\lambda_i(t)}\, d\nu_i(t) + \beta\left[\delta + \sum_{k\neq i}\frac{\lambda_i(t)+\lambda_k(t)}{\lambda_i(t)-\lambda_k(t)}\right] dt \qquad 1 \leq i \leq m.$$

for $\beta = 1, 2$ and $\delta \geq m+1, m$ respectively, where $(\nu_i)_i$ are independent Brownian motions and $\lambda_1(0) > \cdots > \lambda_m(0)$. Recall that the process remains strictly positive if it is initially strictly positive. This suggests to define the β-Laguerre process as the solution, when it exists, of :

$$d\lambda_i(t) = 2\sqrt{\lambda_i(t)}\, d\nu_i(t) + \beta\left[\delta + \sum_{k\neq i}\frac{\lambda_i(t)+\lambda_k(t)}{\lambda_i(t)-\lambda_k(t)}\right] dt \qquad 1 \leq i \leq m, \qquad t < \tau\wedge R_0,$$

where $R_0 = \inf\{t,\, \lambda_m(t) = 0\}$, $\beta, \delta > 0$ and with $\lambda_1(0) > \cdots > \lambda_m(0) > 0$. It is very easy to see that :

$$dD_t := d\left(\prod_{i=1}^m \lambda_i(t)\right) = 2D_t\sqrt{\sum_{i=1}^m \frac{1}{\lambda_i(t)}d\tilde{B}_t} + \beta(\delta - m + 1)D_t\sum_{i=1}^m\frac{1}{\lambda_i(t)}dt$$

for all $t < \tau \wedge R_0$ where $\tilde{B}$ is a standard Brownian motion. It follows that $\forall\, r \in \mathbb{R}$:

$$\begin{aligned} d(\ln(D_t)) &= 2\sqrt{\sum_{i=1}^m \frac{1}{\lambda_i(t)}d\tilde{B}_t} + [\beta(\delta - m) + \beta - 2]\sum_{i=1}^m \frac{1}{\lambda_i(t)}dt \\ d(\det(D_t)^r) &= M_t + r[\beta(\delta - m + 1) + 2r - 2]D_t^r\sum_{i=1}^m\frac{1}{\lambda_i(t)}dt \end{aligned}$$

where $M_t = 2rD_t^r\sqrt{\sum_{i=1}^m 1/\lambda_i(t)d\tilde{B}_t}$. From these two SDE, we can argue as in the Wishart and Laguerre cases that $R_0 > \tau$ a.s. when $\delta \geq m - 1 + 2/\beta$ (choose

$2r = 2 - \beta(\delta - m + 1) < 0$ when $\delta > m - 1 + 2/\beta$ then use McKean's argument). Set $r_i := \sqrt{\lambda_i}$, then, for $t < \tau \wedge R_0$:

$$\begin{aligned} dr_i(t) &= d\nu_i(t) + \frac{1}{2r_i(t)}\left[\beta\delta - 1 + \beta\sum_{j\neq i}\frac{r_i^2 + r_j^2}{r_i^2 - r_j^2}\right]dt \\ &= d\nu_i(t) + \frac{\beta(\delta - m + 1) - 1}{2r_i(t)}dt + \frac{\beta}{2}\sum_{j\neq i}\left[\frac{1}{r_i(t) - r_j(t)} + \frac{1}{r_i(t) + r_j(t)}\right]dt \\ &= d\nu_i(t) + \frac{k_0}{r_i(t)}dt + k_1\sum_{j\neq i}\left[\frac{1}{r_i(t) - r_j(t)} + \frac{1}{r_i(t) + r_j(t)}\right]dt \end{aligned}$$

with $2k_0 = \beta(\delta - m + 1) - 1$, $2k_1 = \beta$. Consequently, the process $r = (r_1, \ldots, r_m)$ defined for all $t < \tau \wedge R_0$ is a B_m-radial Dunkl process. Using Theorem 2.1, one claim that the SDE above has a unique strong solution for all $t \geq 0$ and all β, δ such that k_0, $k_1 > 0$. This strengthen results from matrix theory : in the Wishart setting ($\beta = 1$), the strong uniqueness holds for $\delta > m$ and in the Laguerre case ($\beta = 2$), it holds for $\delta > m - 1/2$. Besides, the generalized Bessel function is given by ([**5**])[4] :

$$\frac{1}{|W|}\sum_{w\in W} D_k(x, wy) = {}_0F_1^{(2/\beta)}(\frac{\beta\delta}{2}, \frac{x^2}{2}, \frac{y^2}{2}) := \sum_{p=0}^{\infty}\sum_{\tau}\left(\frac{\beta\delta}{2}\right)_{\tau}\frac{J_\tau^{(2/\beta)}(x^2/2)J_\tau^{(2/\beta)}(y^2/2)}{J_\tau^{(2/\beta)}(1_m)p!}$$

so that (25) writes

$$(33)\quad p_t^{k_0,k_1}(x, y) = \frac{|W|}{c_k t^{\gamma+m/2}}e^{-(|x|^2+|y|^2)/2t}{}_0F_1^{(2/\beta)}(\frac{\beta\delta}{2}, \frac{x^2}{2t}, \frac{y^2}{2t})\prod_{i=1}^{m}(y_i)^{2k_0}V^{2k_1}(y^2)dy$$

where V stands for the Vandermonde function. Using the variable change $y \mapsto \sqrt{y}$, the semi-group density of the β-Laguerre process writes :

$$q_t^{k_0,k_1}(x, y) = \frac{C_{k_0,k_1}}{t^{\gamma+2k_1+m/2}}e^{-\left(\sum_{i=1}^m(x_i+y_i)/2t\right)}{}_0F_1^{(2/\beta)}(\frac{\beta\delta}{2}, \frac{x}{2t}, \frac{y}{2t})\prod_{i=1}^{m}(y_i)^{k_0-1/2}V^{2k_1}(y)dy$$

For $x = 0$ and $t = 1$, we recover the same p.d.f. given in [**45**] for β-Laguerre ensemble.

Remarks. 1/ Recall that for all $\alpha \in R$, we set $l(\alpha) = k(\alpha) - 1/2$. Hence, in the B_m-case, $l_0 = k_0 - 1/2$, $l_1 = k_1 - 1/2$. For $-l$, all corresponding parameters will be primed. For instance, $-l_0 = k'_0 - 1/2$, $-l_1 = k'_1 - 1/2$. Let us consider a Wishart process of dimension δ' such that $m - 1 \leq \delta' < m + 1$ ([**19**]), $k'_1 = 1/2$ ($\beta' = 1$) and $k'_0 = (\delta' - m)/2 \Rightarrow -l_1 = 0$, $-l_0 = (\delta' - m - 1)/2 < 0$. Set $\delta' = m + 1 - 2\nu$, $0 < \nu <$

[4] With the same notations used in [**5**], one has $\beta a'/2 = k_0$, $a = k_0 - 1/2$, $\beta = 2/\alpha \Rightarrow a + q = \beta\delta/2$.

$1/2$, then, $l_1 = 0, l_1 = \nu \Rightarrow k_1 = 1/2\,(\beta = 1)$ and $k_0 = \nu + 1/2\,(\delta = m + 1 + 2\nu)$. Results of 4.1 writes :

$$P_x^{-l}(T_0 > t) = C_k \prod_{i=1}^{m} \left(\frac{x_i^2}{2t}\right)^{\nu} e^{-|x|^2/2t} {}_1F_1^{(2)}(\frac{m+1}{2}, \frac{\delta}{2}, \frac{x^2}{2t})$$

which fits the expression already derived in [**40**]. When $k_0' = k_1' = 0$ $(-l_0 = -l_1 = -1/2)$, then $k_0 = k_1 = 1(\beta = 2, \delta = m + 1/2)$ and the Jack polynomials fits the Schur functions (see [**86**]). In that case, the following representation holds ([**62**])

$${}_1F_1^{(1)}(a, b, z) = \frac{\det(z_i^{m-j}\,{}_1\mathscr{F}_1(a - j + 1, b - j + 1, z_i)_{1\leq i,j\leq m}}{V(z)}$$

where ${}_1\mathscr{F}_1$ denotes the univariate hypergeometric function. Hence, the tail distribution writes :

$$P_x^{-l}(T_0 > t) = C_k \det\left[\left(\frac{x_i^2}{2t}\right)^{m-j+1/2} e^{-x_i^2/2t} {}_1\mathscr{F}_1(\frac{m+1}{2} - j + 1, m - j + \frac{3}{2}, \frac{x_i^2}{2t})\right]_{1\leq i,j,\leq m}$$

The corresponding process is the Brownian motion in the Weyl chamber of B-type. For Laguerre processes ([**37**]) of dimension δ such that $m - 1/2 \leq \delta < m$, one should apply results derived in section 4.2. Take $k_1' = 1\,(\beta' = 2)$ and $k_0' = \delta' - m + 1/2 \Rightarrow l_1 = 1/2$, $l_0 = \delta' - m := -\nu$ with $0 < \nu < 1/2$. Thus $r_1 = 1/2$, $r_0 = \nu \Rightarrow k_1 = 1\,(\beta = 2)$ and $k_0 = \nu + 1/2\,(\delta = m + \nu)$ so that :

$$P_x^{l}(T_0 > t) = C_k \prod_{i=1}^{m} \left(\frac{x_i^2}{2t}\right)^{\nu} e^{-|x|^2/2t} {}_1F_1^{(1)}(m, \delta, \frac{x^2}{2t})$$

2/Recall that when $\beta = 2$, ${}_0F_1^{(1)}$ has a determinantal representation (see [**62**]) yielding to König and O'Connell result on the V-transform of m-independent squared Bessel processes (BESQs) constrained never to collide (or stay in the A_{m-1}-type Weyl chamber, see [**78**]). Similar results holds for A_{m-1}-type root system with Brownian motions instead of BESQs. Nevertheless, when $k_0 = k_1 = 1\,(\beta = 2, \delta = m + 1/2)$, a similar interpretation involving m- independent Brownian motions killed when they reaches 0 holds. However, the Vandermonde function may be replaced by the product over positive roots. In this case, the eigenvalues process is known as the BM in the Weyl chamber of type B_m (see [**58**] for further details and other root systems). Since $\gamma = m^2$ and from ([**62**], [**37**]) :

$${}_rF_s^{(1)}((m+a_i)_{1\leq i\leq r}, (m+b_j)_{1\leq j\leq s}, x, y) = \frac{\det[{}_r\mathscr{F}_s((a_i + 1)_{1\leq i\leq r}, (b_j + 1)_{1\leq j\leq s}, x_l y_f)]_{l,f}}{V(x)V(y)}$$

(33) transforms to :

$$
\begin{aligned}
p_t^{1,1}(x,y) &= C_m \frac{h(y)}{h(x)} \frac{e^{-(|x|^2+|y|^2)/2t}}{t^{m/2}} \prod_{i,j=1}^{m} \left(\frac{x_i y_j}{t}\right) \det\left[{}_0\mathscr{F}_1\left(\frac{1}{2}+1, \frac{(x_i y_j)^2}{4t^2}\right)\right]_{i,j} \\
&= C_m \frac{h(y)}{h(x)} \frac{e^{-(|x|^2+|y|^2)/2t}}{t^{m/2}} \det\left[\frac{x_i y_j}{t} {}_0\mathscr{F}_1\left(\frac{3}{2}, \frac{(x_i y_j)^2}{4t^2}\right)\right]_{i,j}
\end{aligned}
$$

where h is the product over positive roots. Besides, the following holds (see [**22**]) :

$$
{}_0\mathscr{F}_1(\frac{3}{2}, z) = \frac{C}{2\sqrt{z}} \sinh(2\sqrt{z}).
$$

Thus,

$$
\begin{aligned}
p_t^{1,1}(x,y) &= C_m \frac{h(y)}{h(x)} \frac{1}{(2\pi t)^{m/2}} e^{-(|x|^2+|y|^2)/2t} \det\left[\sinh\left(\frac{x_i y_j}{t}\right)\right]_{i,j} \\
&= \frac{h(y)}{h(x)} \det\left[N_t(y_j - x_i) - N_t(y_j + x_i)\right]_{i,j}
\end{aligned}
$$

where $N_t(u) = (1/\sqrt{2\pi t})e^{-u^2/2t}$, which fits Grabiner's result ([**58**] page 186). This is in agreement with the generator since $\Delta h = 0$ ([**58**]) and

$$
\mathscr{L}f = \Delta f + \Gamma(\log h, f) = \Delta f + \sum_{i=1}^{m} \partial_i(\log h)\partial_i f,
$$

where Γ is the so-called "opérateur du carré du champ" (see [**101**] Chap. VIII). Besides, for $m = 1$, r is a Bessel process of dimension $2\delta = 3$ and the expression inside the determinant in the second line is exactly the semi-group of the Brownian motion killed when it reaches 0 (see [**101**] p. 87).

5.2. Generalized Bessel function in the D_m case. In the classification of root systems, the A_{m-1} and B_m are known to be "irreducible" and both of them correspond to some matrix processes. Another one, yet with no underlying matrices, is the D_m root system defined by (see [**67**] p. 42)

$$
R = \{\pm e_i \pm e_j,\ 1 \le i < j \le m\}, \qquad R_+ = \{e_i \pm e_j,\ 1 \le i < j \le m\}
$$

There is one conjugacy class so that $k(\alpha) = k_1$. Grabiner's result reads for the Brownian motion in the Weyl chamber of D_m-type ($k_1 = 1$) :

$$
\begin{aligned}
p_t^1(x,y) &= \frac{V(y^2)}{V(x^2)} \frac{\det[N_t(y_i - x_j) - N_t(y_i + x_j)] + \det[N_t(y_i - x_j) + N_t(y_i + x_j)]}{2} \\
&= \frac{C_m}{t^{\gamma+m/2}} e^{-(|x|^2+|y|^2)/2t} \frac{\det\left[\sinh(x_i y_j/t)\right] + \det\left[\cosh(x_i y_j/t)\right]}{V(x^2/4t^2)V(y^2)} V^2(y^2)
\end{aligned}
$$

where $\gamma = m(m-1)$. The second term in the sum involves the transition density of a reflected Brownian motion ($|B|$, see [**101**] p. 81). A natural way to interpret the announced formula is that the Weyl chamber is given by :

$$C = \{x \in \mathbb{R}^m,\, x_1 > \cdots > x_{m-1} > |x_m|\},$$

so that C fits the B_m-Weyl chamber when $x_m > 0$, otherwise, it is its conjuguate with respect to s_{e_m} since this simple reflection acts only on x_m and retains the others. With the help of the determinantal formula used before ([**62**], [**37**]), ${}_0\mathscr{F}_1(\frac{3}{2}, z) = C\sinh(2\sqrt{z})/\sqrt{z}$ and ${}_0\mathscr{F}_1(1/2, z) = \cosh(2\sqrt{z})$ ([**22**]), one writes :

$$p_t^1(x,y) = \frac{e^{-(|x|^2+|y|^2)/2t}}{c_k t^{\gamma+m/2}}\left[\prod_{i=1}^{m}\left(\frac{x_iy_i}{2t}\right){}_0F_1^{(1)}\left(m+\frac{1}{2}, \frac{x^2}{2t}, \frac{y^2}{2t}\right) + {}_0F_1^{(1)}\left(m-\frac{1}{2}, \frac{x^2}{2t}, \frac{y^2}{2t}\right)\right]V^2(y^2)$$

With regard to (25) and setting $q = 1 + (m-1)k_1$, it is natural to prove that :

Proposition 5.1.

$$\frac{1}{|W|}\sum_{w\in W} D_k(x,wy) = \prod_{i=1}^{m}\left(\frac{x_iy_i}{2}\right){}_0F_1^{(1/k_1)}\left(q+\frac{1}{2}, \frac{x^2}{2}, \frac{y^2}{2}\right) + {}_0F_1^{(1/k_1)}\left(q-\frac{1}{2}, \frac{x^2}{2t}, \frac{y^2}{2t}\right).$$

Proof : it relies on the following characterization ([**103**]) : given a reduced root system R with finite reflection group W, $D_k^W(x,\cdot)$ is the unique W-invariant function valued 1 at $x=0$ satisfying $\Delta_k D_k^W(x,\cdot) = |x|^2 D_k^W(x,\cdot)$. It is easy to see that the function above is W-invariant since W is the semi-direct product of the symmetric group S_m and $(\mathbb{Z}/2\mathbb{Z})^{m-1}$ acting by even sign changes. However, it is not for the finite reflection group associated to the B_m root system due to the term multiplying the first hypergeometric series. In the D_m case, the Dunkl Laplacian writes on W-invariant functions :

$$\Delta_k = \sum_{i=1}^{m}\partial_i^2 + 2k_1\sum_{i\neq j}\left[\frac{1}{y_i-y_j} + \frac{1}{y_i+y_j}\right]\partial_i$$

Let :

$$\begin{aligned}
f(x,y) &= {}_0F_1^{(1/k_1)}\left(q+\frac{1}{2}, \frac{x^2}{2}, \frac{y^2}{2}\right) \\
g(x,y) &= {}_0F_1^{(1/k_1)}\left(q-\frac{1}{2}, \frac{x^2}{2}, \frac{y^2}{2}\right), \\
d(x,y) &= \prod_{i=1}^{m}(x_iy_i/2)
\end{aligned}$$

considered as functions of the variable y such that the generalized Bessel function is proportional to $df+g$ and $\Delta_k[df+g] = \Delta_k(df) + \Delta_k(g)$. Recall that :

$$\frac{1}{|W|}\sum_{w\in W} D_k^{(B_m)}(x,wy) = {}_0F_1^{(1/k_1)}(k_0 - \frac{1}{2} + q, \frac{x^2}{2}, \frac{y^2}{2})$$

Note also that Δ_k is a particular case of the Dunkl Laplacian considered for the B_m root system when $k_0 = 0$. As a result

$$\Delta_k g(x,y) = \Delta_k^{(B_m)}(k_0 = 0)\left[\frac{1}{|W|}\sum_{w\in W} D_k^{(B_m)}(x, wy)\right] = \frac{1}{|W|} < x, x > g(x,y)$$

For the remaining term, note that both d and f are W-invariant. Write $\Delta_k = \sum_{i=1}^m T_i^2$, where T_i is the difference Dunkl operator (see [**103**] p. 5). Then using (see [**103**] p. 6), one has the derivation formula $T_i(df) = dT_i(f) + fT_i(d)$. It gives that

$$\Delta_k(df) = d\Delta_k(f) + f\Delta_k(d) + 2\sum_{i=1}^m (T_i(d))(T_i(f))$$

Moreover, $T_i(d) = \partial_i(d)$ and $T_i(f) = \partial_i(f)$ by W-invariance. Next we compute :

$$\Delta_k(d)(x,y) = 2k_1 d(x,y)\sum_{i\neq j}\frac{1}{y_i}\left[\frac{1}{y_i - y_j} + \frac{1}{y_i + y_j}\right] = 4k_1 d(x,y)\sum_{i\neq j}\frac{1}{y_i^2 - y_j^2} = 0$$

As a result :

$$\begin{aligned}\Delta_k(df)(x,y) &= [d\Delta_k(f)](x,y) + 2\sum_{i=1}^m [(\partial_i(d))(\partial_i(f))](x,y)\\ &= [d\Delta_k(f)](x,y) + 2[d\sum_{i=1}^m \frac{1}{y_i}(\partial_i(f))](x,y)\\ &= d(x,y)[\Delta_k + 2\sum_{i=1}^m \frac{1}{y_i}\partial_i]f(x,y) = d(x,y)\Delta_k^{(B_m)}(k_0 = 1)f(x,y)\end{aligned}$$

When $k_0 = 1$, f fits the generalized Bessel function in the B_m case $\Rightarrow \Delta_k(df)(x,y) = (1/|W|) < x, x > df(x,y)$. Finally :

$$\Delta_k\left[\frac{1}{|W|}\sum_{w\in W} D_k(x, wy)\right] =< x, x > \left[\frac{1}{|W|}\sum_{w\in W} D_k(x, wy)\right] \qquad \blacksquare$$

6. Alcove-valued process

6.1. β-Jacobi processes. Recall that the eigenvalues of the real Jacobi matrix process of parameters (p, q) (see [**43**] for facts on this process) satisfy :

$$d\lambda_i(t) = 2\sqrt{(\lambda_i(t)(1-\lambda_i(t))}d\nu_i(t) + \left[(p - (p+q)\lambda_i(t)) + \sum_{j\neq i}\frac{\lambda_i(t)(1-\lambda_j(t)) + \lambda_j(t)(1-\lambda_i(t))}{\lambda_i(t) - \lambda_j(t)}\right]dt$$

for $0 < \lambda_m(0) < \cdots < \lambda_1(0) < 1$ and all $t < \inf\{s > 0, \lambda_m(s) = 0 \,\mathrm{or}\, \lambda_1(s) = 1\} \wedge \tau$. The β-Jacobi process is defined as a solution, whenever it exists, of the SDE differing from the one above by a parameter $\beta > 0$ in front of the bracket. It is easy to see that if λ is a β-Jacobi process of parameters (p, q), then $1 - \lambda$ is a

β-Jacobi process of parameters (q,p). As mentioned in the introductory part, the connection with root systems is not new in its own ([**8**]) however we prefer giving some details of this transition. Setting $\lambda_i = \sin^2\phi_i$ then $\phi_i = \arcsin\sqrt{\lambda_i} := s(\lambda_i)$ and $0 < \phi_m < \cdots < \phi_1 < \pi/2$. The first and second derivatives of s are given by :

$$s'(\lambda_i) = \frac{1}{\sin 2\phi_i}, \qquad s''(\lambda_i) = \frac{2(2\sin^2\phi_i - 1)}{\sin^3 2\phi_i} = -\frac{2\cos 2\phi_i}{\sin^3 2\phi_i}$$

Using

$$\sin^2\phi_i - \sin^2\phi_j = 2\sin(\phi_i+\phi_j)\sin(\phi_i-\phi_j)$$

$$\sin^2\phi_i\cos^2\phi_j + \cos^2\phi_i\sin^2\phi_j = \sin^2(\phi_i+\phi_j) + \sin^2(\phi_i-\phi_j)$$

then, Ito's formula gives :

$$\begin{aligned} d\phi_i(t) &= d\nu_i(t) + \beta\frac{(p-(p+q)\sin^2\phi_i)}{\sin 2\phi_i} - \cot 2\phi_i dt \\ &+ \frac{\beta}{2}\frac{dt}{\sin 2\phi_i(t)}\sum_{j\neq i}\frac{\sin^2(\phi_i(t)+\phi_j(t)) + \sin^2(\phi_i(t)-\phi_j(t))}{\sin(\phi_i(t)+\phi_j(t))\sin(\phi_i(t)-\phi_j(t))} \end{aligned}$$

Writing $\sin^2\phi_i = (1-\cos 2\phi_i)/2$, then

$$\begin{aligned} d\phi_i(t) &= d\nu_i(t) + \beta\frac{(p-q)}{2}\frac{dt}{\sin 2\phi_i(t)} + \frac{\beta(p+q)-2}{2}\cot 2\phi_i(t)dt \\ &+ \frac{\beta}{2}\frac{dt}{\sin 2\phi_i(t)}\sum_{j\neq i}\frac{\sin^2(\phi_i(t)+\phi_j(t)) + \sin^2(\phi_i(t)-\phi_j(t))}{\sin(\phi_i(t)+\phi_j(t))\sin(\phi_i(t)-\phi_j(t))} \end{aligned}$$

where $0 < \phi_m(0) < \cdots < \phi_1(0) < \pi/2$. Moreover,

$$\sin 2\phi_i = [\cot(\phi_i+\phi_j) + \cot(\phi_i-\phi_j)]\sin(\phi_i+\phi_j)\sin(\phi_i-\phi_j)$$

which gives

$$\begin{aligned} d\phi_i(t) &= d\nu_i(t) + \beta\frac{(p-q)}{2}\frac{dt}{\sin 2\phi_i(t)} + \frac{\beta(p+q)-2}{2}\cot 2\phi_i(t)dt \\ &+ \frac{\beta}{2}\sum_{j\neq i}\frac{[1/\sin^2(\phi_i(t)+\phi_j(t))] + [1/\sin^2(\phi_i(t)-\phi_j(t))]}{\cot(\phi_i(t)+\phi_j(t)) + \cot(\phi_i(t)-\phi_j(t))}dt \end{aligned}$$

Using $1+\cot^2 z = 1/\sin^2 z$, then

$$d\phi_i(t) = d\nu_i(t) + \beta\frac{(p-q)}{2}\frac{dt}{\sin 2\phi_i(t)} + \frac{\beta(p+q)-2}{2}\cot 2\phi_i(t)dt$$
$$+\frac{\beta}{2}\sum_{j\neq i}\frac{\cot^2(\phi_i(t)+\phi_j(t))+\cot^2(\phi_i(t)+\phi_j(t))+2}{\cot(\phi_i(t)+\phi_j(t))+\cot(\phi_i(t)-\phi_j(t))}dt$$
$$= d\nu_i(t) + \beta\frac{(p-q)}{2}\frac{dt}{\sin 2\phi_i(t)} + \frac{\beta(p+q)-2}{2}\cot 2\phi_i(t)dt + \beta\times$$
$$\sum_{i\neq j}\left\{\frac{1-\cot(\phi_i(t)+\phi_j(t))\cot(\phi_i(t)-\phi_j(t))}{\cot(\phi_i(t)+\phi_j(t))+\cot(\phi_i(t)-\phi_j(t))} + \frac{\cot(\phi_i(t)+\phi_j(t))+\cot(\phi_i(t)-\phi_j(t))}{2}\right\}dt$$

Using

$$-\cot(u+v) = \frac{1-\cot(u)\cot(v)}{\cot(u)+\cot(v)}.$$

and

$$\frac{1}{\sin 2\phi_i} = \frac{2\cos^2\phi_i - \cos 2\phi_i}{2\sin\phi_i\cos\phi_i} = \cot\phi_i - \cot 2\phi_i$$

we finally obtain

(34)

$$d\phi_i(t) = d\nu_i(t) + \left[k_0\cot\phi_i + k_1\cot 2\phi_i(t)dt + k_2\sum_{i\neq j}[\cot(\phi_i+\phi_j)+\cot(\phi_i-\phi_j)]\right]dt$$

where

$$2k_0 = \beta(p-q), \quad k_1 = \beta(q-(m-1))-1, \quad 2k_2 = \beta. \tag{35}$$

Easy computations show that $\pi/2-\phi$ satisfies (34) with (p,q) intertwined.

6.2. Eigenfunctions and Heckman-Opdam's functions. Let $k_2 > 0$ and $\mathscr{L}$ be the generator of ϕ, then the eigenfunctions of $\mathscr{L}$ are given by Gauss hypergeometric series : in fact, let $\mathscr{A}$ be the generator of $(\lambda_1,\dots,\lambda_m)$ (see [**43**] p. 135 for $\beta = 1$) :

$$\mathscr{A} = 2\sum_{i=1}^m \lambda_i(1-\lambda_i)\partial_i^2 + \beta\sum_{i=1}^m\left[p-(p+q)\lambda_i + \sum_{j\neq i}\frac{\lambda_i(1-\lambda_j)+\lambda_j(1-\lambda_i)}{\lambda_i-\lambda_j}\right]\partial_i$$
$$= 2\sum_{i=1}^m \lambda_i(1-\lambda_i)\partial_i^2 + \beta\sum_{i=1}^m[p-(m-1)-(p+q-2(m-1))\lambda_i]\partial_i + 2\beta\sum_{i\neq j}\frac{\lambda_i(1-\lambda_i)}{\lambda_i-\lambda_j}\partial_i$$

From Equation (32) (k_2 plays the role of k_1), one can see that ${}_2F_1^{(1/k_2)}(a,b,c;\lambda)$ is the unique symmetric analytic function u such that $u(0)=1$ which satisfies

$$\mathscr{A}u(\lambda) = 2mab\,u(\lambda), \quad 2c = \beta p = 2k_0 + k_1 + 2k_2(m-1)+1,\ 2a+2b+1-2c = k_1.$$

with k_i, $0 \leq i \leq 2$ cited in (35). Setting $\sin^2 \phi := (\sin^2 \phi_1, \ldots, \sin^2 \phi_m)$, then $\mathscr{A}$ transforms to $\mathscr{L}$. Hence :

$$\mathscr{L}[u(\sin^2 \phi)] = 2mab\,[u(\sin^2 \phi)]$$

In the same spirit, one can also interpret $\mathscr{L}$ as the "radial part" of the trigonometric version Dunkl-Cherednik Laplacian (with cot replacing coth, [**8**],[**92**]). By radial part, we mean the restriction on W-invariant functions. Besides, this Laplacian arises, as for Dunkl and Cherednik-Dunkl ones, from differential-difference first-order operators. However, this comes beyond the spirit of this work and will not be done here.

6.3. Existence and uniqueness of a strong solution. The involved root system is the non reduced BC_m defined by

$$\begin{aligned} R &= \{\pm e_i,\, \pm 2e_i,\, 1 \leq i \leq m,\, \pm(e_i \pm e_j),\, 1 \leq i < j \leq m\} \\ R_+ &= \{e_i,\, 2e_i,\, 1 \leq i \leq m,\, (e_i \pm e_j),\, 1 \leq i < j \leq m\} \\ \Delta &= \{e_i - e_{i+1},\, 1 \leq i \leq m-1,\, e_m\} \end{aligned}$$

When $k_0 = 0 (p = q)$, it reduces to the reduced C_m system

$$\begin{aligned} R &= \{\pm e_i \pm e_j,\, 1 \leq i < j \leq m,\, \pm 2e_i,\, 1 \leq i \leq m\} \\ R_+ &= \{e_i \pm e_j,\, 1 \leq i < j \leq m,\, 2e_i,\, 1 \leq i \leq m\} \\ \Delta &= \{e_i - e_{i+1},\, 1 \leq i \leq m-1,\, 2e_m\} \end{aligned}$$

and it is known as the ultraspheric case. The Weyl group action on $\mathbb{R}^m$ gives rise to three orbits so that the multiplicity function is given by $k = (k_0, k_1, k_2)$. Setting $\tilde{\phi}_i := \phi_i/\pi$, then the process is valued in the *positive Weyl alcove* (see [**67**]) defined by :

$$\tilde{A} = \{\tilde{\phi} \in \mathbb{R}^m,\, < \alpha, \tilde{\phi} >> 0\, \forall \alpha \in \Delta\; < \tilde{\alpha}, \tilde{\phi} >< 1\}$$

where $\tilde{\alpha} = 2e_1$ is the highest positive root (that is $\tilde{\alpha} - \alpha \in R_+ \,\forall \alpha \in R$, see [**67**]). The associated affine Weyl group W_a is the semi-direct product of W and the translation group corresponding to the coroot lattice ($\mathbb{Z}$-span of $\{2\alpha/||\alpha||^2,\, \alpha \in R\}$). The generator writes in this case :

$$\mathscr{L}g(\phi) := \frac{1}{2}\Delta g(\phi) - < \nabla g(\phi), \nabla \Phi(\phi) >, \quad \Phi(\phi) = -\sum_{\alpha \in R_+} k(\alpha) \log \sin(< \alpha, \phi >)$$

Thus, with minor modifications, Theorem 2.1 states that (34) has a unique strong solution for all $t > 0$ subject to $k_0 > 0,\, k_1 > 0,\, k_2 > 0 \Leftrightarrow \beta > 0,\, p > q > (m-1) + 1/\beta$. Applying this to $\pi/2 - \phi$, this holds for $\beta > 0$, $q > p > (m-1)+1/\beta$. Since the ultraspheric case still involves a root system, then (34) has a unique strong solution for $p \wedge q > (m-1) + 1/\beta$ which simplifies to $p \wedge q > m$ in the real case $\beta = 1$ and $p \wedge q > m - 1/2$ in the complex one $\beta = 2$. Theorem 2.1 is modified as follows : $\partial \tilde{A} = \cup_{\alpha \in \Delta} H_\alpha \cup H_{\alpha,1}$ where

$$H_{\alpha,1} = \{\tilde{\phi},\, < \tilde{\alpha}, \tilde{\phi} >= 1\} = \{\phi,\, \pi - < \tilde{\alpha}, \phi >= 0\}$$

Compared with (28), the convex function $x \mapsto -\ln(<\alpha, x>)$ should be substituted by $\phi \mapsto -\ln(\sin(<\alpha, \phi>))$ and one has to deal with an additional term in the expression of the boundary process $(L_t)_{t\geq 0}$: $\mathbf{1}_{\{\pi - <\tilde{\alpha},\phi>=0\}}$. Then the occupation density formula writes :

$$\begin{aligned}\int_0^{\pi/2} L_t^a(\pi - <\tilde{\alpha}, \phi>)|\theta^{'}(a)|da &= <\tilde{\alpha}, \tilde{\alpha}> \int_0^t |\theta^{'}(\pi - <\tilde{\alpha}, X_s>)|ds \\ &= <\tilde{\alpha}, \tilde{\alpha}> \int_0^t |\theta^{'}(<\tilde{\alpha}, X_s>)|ds\end{aligned}$$

since $\cot(\pi - z) = -\cot(z)$. Hence, the same proof applies and Lemma 1 remains valid for $\tilde{\alpha} \in R_+$. Besides, either it will exist $\alpha \in \Delta$ such that $<\alpha, x>= 0$ and Lemma 2.2 applies, or we will need to prove that $<n(x), \tilde{\alpha}>\neq 0$ if x belongs only to $H_{\tilde{\alpha},1}$. Let us first recall that the highest root is the unique positive root such that $\tilde{\alpha} - \alpha \in R_+$ for all $\alpha \in R_+$. Thus it may be written as $\tilde{\alpha} = \sum_{\alpha\in\Delta} a_\alpha \alpha$ where $a_\alpha \geq 1$. Else, if there exists $\alpha_0 \in \Delta$ such that $a_{\alpha_0} < 1$ and since $\tilde{\alpha}$ must be greater than all simple roots (in particular greater than α_0) then

$$\tilde{\alpha} - \alpha_0 = (a_{\alpha_0} - 1)\alpha_0 + \sum_{\alpha_0\neq\alpha\in\Delta} a_\alpha \alpha = c_{\alpha_0}\alpha_0 + \sum_{\alpha_0\neq\alpha\in\Delta} c_\alpha \alpha$$

for some $c_\alpha \geq 0$. Our claim follows from the fact that Δ is a basis. Next, it is not difficult to see from the definition of $n(x)$ and the fact that $<\alpha, x>> 0$ for all $\alpha \in \Delta$ that $n(x)$ is colinear to $-\sum_{\alpha\in\Delta}\alpha$. It follows that

$$<n(x), \tilde{\alpha}>= -c \sum_{\alpha\in\Delta} <\alpha, \tilde{\alpha}>= -c \sum_{\alpha\in\Delta}\sum_{\theta\in\Delta} a_\alpha <\alpha, \theta>$$

If $<n(x), \tilde{\alpha}>= 0$, then

$$||\sum_{\alpha\in\Delta}\alpha||^2 = \sum_{\alpha\in\Delta}\sum_{\theta\in\Delta} <\alpha, \theta>\leq \sum_{\alpha\in\Delta}\sum_{\theta\in\Delta} a_\alpha <\alpha, \theta>= 0$$

which implies that $n(x) = 0$. ■

6.4. Brownian motion in Weyl alcoves. Let

$$h_1(\phi) := \prod_{\alpha\in R_+} \sin(<\alpha, \phi>)$$

Then, h_1 is strictly positive on $\tilde{A}$ and vanishes for $\phi \in \partial A$. One can also show that $(1/2)\Delta h_1 = ch$ for some strictly negative constant c. Let $P_t^{h_1}$ denote the semi group given by :

$$P_t^{h_1} f(\phi) := e^{-ct}\frac{P_t(h_1 f)(\phi)}{h_1(\phi)},$$

where P_t denotes the semi group of the process consisting of m-independent BMs in A killed when it first reaches ∂A. The corresponding generator writes :

$$\mathscr{L}^{h_1}(f) = \frac{1}{h_1}\left[\frac{1}{2}\Delta - c\right](h_1 f) = \frac{1}{2}\Delta + \sum_{i=1}^{m}(\partial_i \log h_1)\partial_i f$$

which fits our generator for $k_2 = 1\,(\beta = 2), k_1 = 2\,(q = m + 1/2)$, $k_0 = 1(p = q + 1 = m + 3/2)$. In the ultraspheric case, this becomes $\beta = 2$, $p = q = m + 1/2$. In both cases, these parameters correspond to the process consisting of m BMs constrained to stay in the BC_m and C_m- Weyl alcoves respectively. Note that p, q are not integers which means that these processes BM can not be realized as eigenvalues processes of complex matrix Jacobi processes which is also the case for the BM in the B_m-Weyl chamber since $\delta = m + 1/2$.

6.5. The first hitting time $\tilde{T}_0$. We define similarly the first hitting time of alcove's walls by $\tilde{T}_0 = \inf\{t > 0,\, (\phi(t)/\pi) \in \partial\tilde{A}\} = \tilde{T}_{\tilde{\alpha}} \wedge \inf\{\tilde{T}_\alpha,\, \alpha \in \Delta\}$, where

$$\begin{aligned}\tilde{T}_\alpha &:= \inf\{t > 0,\, < \alpha, \phi(t) >= 0\},\\ \tilde{T}_{\tilde{\alpha}} &:= \inf\{t > 0, < \tilde{\alpha}, \phi(t) >= 2\phi_1 = \pi\},\end{aligned}$$

and ϕ is the unique strong solution for all $t \geq 0$ of [5] :

$$d\phi(t) = d\nu(t) + \sum_{\alpha \in R+} k(\alpha)\cot(< \alpha, \phi(t) >)\alpha\, dt, \qquad \frac{\phi(0)}{\pi} \in \tilde{A},$$

for the non reduced root system $R = BC_m$ with $k(\alpha) > 0$ for all α and $p \wedge q > (m-1) + 1/\beta$. Let us focus on $\tilde{T}_{\alpha_0}$ for some $\alpha_0 \in \Delta$. We shall distinguish two cases :

6.5.1. $\alpha_0 = e_i - e_{i+1}$, $1 \leq i \leq m-1$. The same scheme described in the proof of Proposition 3.1 applies here since the main ingredients used there are the SDE and the fact that $\sigma_0(\alpha) \in R_+$ if $\alpha \neq \alpha_0$. The second assertion follows from $\sigma_0(2e_j) = 2\sigma_0(e_j) = 2(\delta_{ij}e_{i+1} + \delta_{(i+1)j}e_i + \mathbf{1}_{\{j\neq i, j\neq i+1\}}e_j) \in R_+$. As a result, one writes for all $t \geq 0$:

$$d < \alpha_0, \phi(t) >= ||\alpha_0||d\gamma_t + k_2||\alpha_0||^2 \cot < \alpha_0, \phi(t) > dt + \sum_{\substack{\alpha \in R_+\\ \alpha \neq \alpha_0}} k(\alpha)a(\alpha)\cot < \alpha, \phi(t) > dt$$

where $a(\alpha) =< \alpha_0, \alpha >$.

$$d < \alpha_0, \phi(t) >= ||\alpha_0||d\gamma_t + k_2||\alpha_0||^2 \cot(< \alpha_0, \phi(t) >)dt + F_t$$

where

$$F_t = \sum_{\substack{\alpha \in R_+ \setminus \alpha_0\\ a(\alpha) > 0}} k(\alpha)a(\alpha)[\cot(< \alpha, \phi(t) >) - \cot(< \sigma_0(\alpha), \phi(t) >)],$$

[5] $k(2e_i) = k_1/2$ for all $1 \leq i \leq m$.

where $\sigma_0 = \sigma_{\alpha_0}$. This drift is strictly negative on $\{\tilde{T}_{\alpha_0} = \infty\}$ since $\phi \mapsto \cot\phi$ is a decreasing function, $< \alpha_0, \phi(t) >> 0$ and since :

$$< \alpha - \sigma_0(\alpha), \phi(t) >= 2\frac{a(\alpha)}{||\alpha||^2} < \alpha_0, \phi(t) >> 0.$$

This implies that $\mathbb{P}_x(\forall t \geq 0, < \alpha_0, \phi(t) >\leq Z_t) = 1$ where $\phi(0) = x$ and :

$$dZ_t = ||\alpha_0||d\gamma_t + ||\alpha_0||^2 k_2 \cot(Z_t)dt, \quad Z_0 =< \alpha_0, \phi(0) >= x$$

on the same probability space. Using (34) with $\beta = 1, m = 1$, one can easily see that $(Z_t)_{t\geq 0} = (\arcsin\sqrt{J}_{||\alpha_0||^2 t})_{t\geq 0}$ where J is a one dimensional Jacobi process of parameters $d = 2k_2 + 1, d' = 1$ (see [**112**]) : that is :

$$dJ_t = 2\sqrt{J_t(1-J_t)}d\gamma_t + (d - (d+1)J_t)dt, \quad 0 < k_2 < 1/2 \Leftrightarrow 0 < d < 2.$$

As J hits 0 almost surely when $0 < d < 2$ (use the skew product in [**112**] and properties of squared Bessel processes), then so does Z and by the way $< \alpha_0, \phi >$ for $k_2 < 1/2 \Rightarrow \tilde{T}_{\alpha_0} < \infty$ a. s.

6.5.2. $\alpha_0 = e_m$. Compared with the previous case, the difference arises from the fact that $\sigma_0(\alpha) \in R_+$ if $\alpha \in R_+ \setminus \{e_m, 2e_m\}$ and the latter is easily checked since for $\alpha = e_i \pm e_j$ this amounts to consider the reduced root system B_m, else for $\alpha = e_i$ with $i \neq m$, $\sigma_0(e_i) = e_i$. According to this, one gets :

$$d < \alpha_0, \phi(t) >= d\phi_m(t) = d\gamma_t + k_0\cot(\phi_m(t))dt + k_1\cot(2\phi_m) + F_t$$

where

$$F_t = \sum_{\substack{\alpha\in R_+\setminus\{e_m,2e_m\} \\ a(\alpha)>0}} k(\alpha)a(\alpha)[\cot(< \alpha, \phi(t) >) - \cot(< \sigma_0(\alpha), \phi(t) >)]$$

where $R_+^1 = \{e_i - e_j,\, 1 \leq i < j \leq m\}$. Using once again (34), we shall compare this process with $(\arcsin\sqrt{J_t})_{t\geq 0}$ where

$$dJ_t = 2\sqrt{J_t(1-J_t)}d\gamma_t + (d - (d+d')J_t)dt, \quad d' = k_1 + 1,\, d = 2k_0 + k_1 + 1.$$

Hence, $\tilde{T}_{e_m} < \infty$ a.s. if $0 < 2k_0 + k_1 < 1/2 \Leftrightarrow \beta p - (\beta(m-1) < 2$. This agrees with the case $m = 1$ for which $p < 2$ (use the skew product in [**112**]). Finally, note that since $a(\alpha) = 0$ for $\alpha \in \{e_i, 2e_i,\, 1 \leq i \leq m-1\}$, F only involves $k_2 = \beta$ which is independent from p, q. Keeping in mind that $\pi/2 - \phi$ is still a β-Jacobi process with (p, q) intertwined which has no effect on the strict negativity of F by the above remark, we conclude that $\tilde{T}_{\tilde{\alpha}} < \infty$ for $0 < \beta q - \beta(m-1) < 2$. ■

6.6. Semi-group density. We end this paper by giving the semi group density of the β-Jacobi process. Before proceeding, we briefly consider two cases for which we can write down the semi-group density : the univariate case and the complex Hermitian one ($\beta = 2$). Let $P_n^{r,s}$ denote the Jacobi polynomial of degree n defined by ([**3**]) :

$$P_n^{r,s}(\lambda) := \frac{(r+1)_n}{n!} {}_2F_1\left(-n, n+r+s+1, r+1; \frac{1-\lambda}{2}\right),$$

for $\lambda \in [-1,1]$, $r, s > -1$, where ${}_2F_1$ is the univariate Gauss hypergeometric function. These polynomials are orthogonal with respect to the measure $Z^{r,s}(\lambda)d\lambda := (1-\lambda)^r(1+\lambda)^s d\lambda$ and the associated inner product in $L^2([-1,1])$ given by

$$< f, g >_{L^2([-1,1])} := \int_{[-1,1]} f(\lambda)g(\lambda)Z^{r,s}(\lambda)d\lambda$$

Moreover, $(P_n^{r,s})_{n\geq 0}$ form a complete set of this Hilbert space and satisfy

$$\left\{\sqrt{1-\lambda^2}\partial_\lambda^2 + [(s-r) - (s+r+2)\lambda]\partial_\lambda\right\} P_n^{r,s}(\lambda) = -n(n+r+s+1)P_n^{r,s}(\lambda)$$

The above eigenoperator defines a diffusion which is related to the one we considered with $m = 1$ via the map $\lambda \mapsto (1-\lambda)/2$ and a deterministic time change $(t \mapsto t/2)$. The semi group density w.r.t Lebesgue measure is written (see [**114**])

$$p_t^{r,s}(\theta, \lambda) = \sum_{n=0}^{\infty} e^{-2r_n t} P_n^{r,s}(\theta)P_n^{r,s}(\lambda)W^{r,s}(\lambda)$$

where r_n denotes the eigenvalues above, $(P_n^{r,s})_n$ are orthonormal polynomials, $p = 2(r+1), q = 2(s+1)$ and $W^{r,s}(\lambda)d\lambda$ is the probability measure corresponding to the measure $Z^{r,s}(\lambda)d\lambda$. No closed forms seems to be known for this density, nonetheless an attempt to get a handier expression was tried in [**?**]. Multivariate analogs appeared in literature ([**8**], [**70**], [**81**] for instance) and are obtained by applying the Gram-Schmidt orthogonalization to the symmetric Jack polynomials w.r.t. measure

$$Z_m^{r,s,\beta}(\lambda)d\lambda := \prod_{i=1}^{m} \lambda_i^r(1-\lambda_i)^s \prod_{1\leq i<j\leq m} |\lambda_i - \lambda_j|^\beta d\lambda_1 \ldots d\lambda_m$$

We shall denote them[6] by $P_\tau^{r,s,\beta}$ for a given partition τ (instead of $G_\tau^{\alpha,\beta}$ used in literature) and stress that some of the properties cited above extend to the higher dimensional case ([**81**]) : an expansion in terms of ${}_2F_1^{(2/\beta)}(-l, b, c, \lambda)$ exists for $\tau = (l^m)$ with m components all equal to l ; $(P_\tau^{r,s,\beta})$, where τ is a partition of length $\leq m$, form a basis of the Hilbert space $L^2([0,1]^m, W_m^{r,s,\beta}(\lambda)d\lambda)$ where $W_m^{r,s,\beta}(\lambda)d\lambda$ is the normalization $Z_m^{r,s,\beta}(\lambda)d\lambda$ in order to be a probability measure ([**81**]). The normalizing constant is given by a McDonald-Selberg integral computed in [**70**]. Moreover, $(P_\tau^{r,s,\beta})_\tau$ are the unique symmetric polynomial eigenfunctions of the

[6]The normalization is different from the one used in both [**8**] and [**81**].

Laplace Beltrami operator $-\mathscr{L}$ (thus defined on $[0,1]^m$) with $\beta(p-(m-1)) = 2(r+1)$, $\beta(q-(m-1)) = 2(s+1)$, associated with the eigenvalues

$$(36)\quad 2r_{n,\tau}^{\beta} := 2\left[\sum_{i=1}^{m}\tau_i(\tau_i - 1 - \beta(i-1)) + |\tau|(r+s+\beta(m-1)+2)\right],\ |\tau| = n.$$

However, with regard to the strong uniqueness for all $t \geq 0$ previously derived, we shall restrict ourselves to $p \wedge q > (m-1) + 1/\beta$. $\beta(q-(m-1)) > 1$ is equivalent to $s > -1/2$ and $\beta(p-(m-1)) > 1$ is equivalent to $r > -1/2$. As a result, $r, s > -1/2$.
It is known that the eigenvalues process of the complex Hermitian Jacobi process (or 2-Jacobi process) is the h-transform (in the Doob sense) for $h = V$ of a process whose components are real Jacobi processes of parameters $2(p-(m-1)) = 2(r+1), 2(q-(m-1)) = 2(s+1)$ constrained to never collide (or to stay in the A_{m-1}-type Weyl chamber). Here, V denotes as usual the Vandermonde function. More precisely, V is an eigenfunction of the generator of the one dimensional Jacobi process of parameters (p,q) (see appendix in [**43**]), say $\mathbf{L}$, that is

$$\mathbf{L}V = cV = -m(m-1)\left(\frac{2(m-2)}{3} + \frac{p+q}{2}\right)V$$

Noting that the parameters r, s are the same both in the univariate and in the multivariate cases, it follows by Karlin-McGregor formula ([**71**]) that the semi group density writes on $\{0 < \lambda_m < \cdots < \lambda_1 < 1\}$

$$\begin{aligned}
K_t^{r,s,2}(\theta,\lambda) &:= e^{-ct}\frac{V(\lambda)}{V(\theta)}\det\left(\sum_{n=0}^{\infty} e^{-2n(n+r+s+1)t} P_n^{r,s}(\theta_i)P_n^{r,s}(\lambda_j)W^{r,s}(\lambda_j)\right)_{i,j} \\
&= e^{-ct}\det\left(\sum_{n=0}^{\infty} e^{-2n(n+r+s+1)t} P_n^{r,s}(\theta_i)P_n^{r,s}(\lambda_j)\right)_{i,j}\frac{W_m^{r,s,2}(\lambda)}{V(\theta)V(\lambda)} \\
&= e^{-ct}\left[\sum_{\sigma_1\in S_m}\epsilon(\sigma_1)\sum_{n_1,\dots,n_m\geq 0} e^{-2\sum_{i=1}^m n_i(n_i+r+s+1)t}\prod_{i=1}^m P_{n_i}^{r,s}(\theta_i)P_{n_i}^{r,s}(\lambda_{\sigma_1(i)})\right]\frac{W_m^{r,s,2}(\lambda)}{V(\theta)V(\lambda)} \\
&= e^{-ct}\left[\sum_{\sigma_1,\sigma_2\in S_m}\epsilon(\sigma_1)\sum_{n_1\geq\cdots\geq n_m\geq 0} e^{-2\sum_{i=1}^m n_{\sigma_2(i)}(n_{\sigma_2(i)}+r+s+1)t}\prod_{i=1}^m P_{n_{\sigma_2(i)}}^{r,s}(\theta_i)P_{n_{\sigma_2(i)}}^{r,s}(\lambda_{\sigma_1(i)})\right]\frac{W_m^{r,s,2}(\lambda)}{V(\theta)V(\lambda)}
\end{aligned}$$

Note that, for a given partition $(n_1 \geq \cdots \geq n_m \geq 0)$ and a permutation $\sigma_2 \in S_m$, one has

$$\sum_{i=1}^m n_{\sigma_2(i)}(n_{\sigma_2(i)} + r + s + 1) = \sum_{i=1}^m n_i(n_i + r + s + 1)$$

Thus summing first over σ_1 with the change of variables $\sigma = \sigma_1\sigma_2$, one gets :

$$K_t^{r,s,2}(\theta,\lambda) = e^{-ct}\sum_{n_1\geq\cdots\geq n_m\geq 0} e^{-2\sum_{i=1}^m n_i(n_i+r+s+1)t}\frac{\det[P_{n_i}^{r,s}(\theta_j)]_{i,j}}{V(\theta)}\frac{\det[P_{n_i}^{r,s}(\lambda_j)]_{i,j}}{V(\lambda)}W_m^{r,s,2}(\lambda)$$

$$= e^{-ct}\sum_{n_1>\cdots>n_m\geq 0} e^{-2\sum_{i=1}^m n_i(n_i+r+s+1)t}\frac{\det[P_{n_i}^{r,s}(\theta_j)]_{i,j}}{V(\theta)}\frac{\det[P_{n_i}^{r,s}(\lambda_j)]_{i,j}}{V(\lambda)}W_m^{r,s,2}(\lambda)$$

Set $n_i = \tau_i + m - i$, then $\tau_1 >\geq \cdots \geq \tau_m \geq 0$. Moreover, with regard to (36), one easily check that

$$r_{n,\tau}^2 = \sum_{i=1}^m \tau_i(\tau_i + r + s + 1 + 2(m-i))$$

so that

$$\sum_{i=1}^m n_i(n_i + r + s + 1) = r_{n,\tau}^2 - c/2$$

The final result writes

$$K_t^{r,s,2}(\theta,\lambda) = \sum_{\tau_1\geq\ldots\tau_m\geq 0} e^{-2r_{n,\tau}^2}\frac{\det[P_{\tau_i+m-i}^{r,s}(\theta_j)]_{i,j}}{V(\theta)}\frac{\det[P_{\tau_i+m-i}^{r,s}(\lambda_j)]_{i,j}}{V(\lambda)}W_m^{r,s,2}(\lambda)$$

$$= \sum_{\tau_1\geq\ldots\tau_m\geq 0} e^{-2r_{n,\tau}^2}P_\tau^{r,s,2}(\theta)P_\tau^{r,s,2}(\lambda)W_m^{r,s,2}(\lambda)$$

where we used the determinantal representation of the Jacobi multivariate polynomials in the complex case[7] (see [**81**]) :

$$P_\tau^{r,s,2}(\lambda) = \frac{\det[P_{\tau_i+m-i}^{r,s}(\lambda_j)]_{i,j}}{V(\lambda)}$$

From these observations, it is natural to claim that :

Proposition 6.1. *The semi group density of the β-Jacobi process is given by*

$$K_t^{r,s,\beta}(\theta,\lambda) := \sum_{n=0}^\infty\sum_{|\tau|=n} e^{-r_{n,\tau}t}P_\tau^{r,s,\beta}(\theta)P_\tau^{r,s,\beta}(\lambda)W_m^{r,s}(\lambda)\mathbf{1}_{\{0<\lambda_m<\cdots<\lambda_1<1\}} \tag{37}$$

with respect to $d\lambda$. As a result, it is positive.

Proof : given a bounded symmetric function f on $[0,1]^m$, define

$$T_tf(\theta) := \int_{0<\lambda_m<\cdots<\lambda_1<1} f(\lambda)\sum_{n=0}^\infty\sum_{|\tau|=n} e^{-2r_{n,\tau}^\beta t}P_\tau^{r,s,\beta}(\theta)P_\tau^{r,s,\beta}(\lambda)W_m^{r,s}(\lambda)d\lambda$$

for $\theta = (0 < \theta_1 < \cdots < \theta_m < 1)$ and $T_0f = f$. The above expression makes sense : this uses the boundness of f, the exponential term with strictly positive t and Fubini Theorem. Besides, $T_t\mathbf{1} = 1$ and $||T_t||$ is bounded for all $t \geq 0$. The first

[7]We adopt a different normalization since we consider orthonormal polynomials.

claim follows easily from the orthogonality of the Jacobi polynomials and $P_0 = \mathbf{1}$ so that the only non zero term is that correponding to $n = 0$. The second one is obvious for $t = 0$ and uses the exponential term when $t \geq \epsilon > 0$. One also easily checks that $T_t T_s = T_{t+s}$ and that $\mathscr{L} T_t f(\lambda) = \partial_t T_t f(\lambda)$ using the dominated convergence theorem. Now, let us consider the Cauchy problem associated to $\mathscr{L}$:

$$\begin{cases} \dfrac{\partial u_f}{\partial t}(t,\lambda) = \mathscr{L} u_f(t,\lambda) \\ u_f(0,\cdot) = f, \end{cases}$$

where $u_f \in C^{1,2}(\mathbb{R}_+^\star \times S := \{0 < \lambda_m < \cdots < \lambda_1 < 1\}) \cap C_b(\mathbb{R}^+ \cap S)$ with reflecting boundary condition :

$$< \nabla u(t,\lambda), n(\lambda) >= 0, \quad (t,\lambda) \in \mathbb{R}_+^\star \times \partial S$$

where $n(\lambda)$ is a unitary inward normal vector at λ. Define $u_t(f)(\lambda) := u_f(t,\lambda)$. It is known ([**108**]) that the above Cauchy problem has a unique solution. Consequently, $(T_t)_{t\geq 0}$ is the semi group of the eigenvalues process $(\lambda(t))_{t\geq 0}$ with density given by $K_t^{r,s,\beta}$. ■

Remark. As the reader can check, the computations performed in the complex Hermitian case do not restrict to Jacobi polynomials. We only used the determinantal representation in terms of their univariate counterparts. As a result, one gets similar formulas replacing Jacobi by Hermite and Laguerre polynomials.

Now, we are able to answer some open questions left in [**43**]. For the real Jacobi matrix ($\beta = 1$), it is known that for $p \wedge q \geq m-1$ and if the eigenvalues are distinct at time $t = 0$, then they remain distinct forever. It is then natural to wonder if this remains valid when starting from non distinct eigenvalues (see [**43**] p. 138-139). The Markov property together with the previous result for distinct eigenvalues are sufficient to claim that this is true provided that the eigenvalues semi group has a density which is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^m$. By virtue of $K_t^{r,s,1}(\theta,\phi)$, for $p \wedge q > m$,

$$\mathbb{P}_{\lambda(0)}(\forall t \geq 0, \forall i \neq j, \lambda_i(t) \neq \lambda_j(t)) = 1, \quad \lambda_1(0) \geq \cdots \geq \lambda_m(0).$$

We argue in the same way to claim that for $p \wedge q \geq m+1$, the process will never hit the boundaries (0 and 1 for λ or 0 and $\pi/2$ for ϕ) even if it did at time $t = 0$.
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CHAPITRE 5

β-Hermitian Brownian motion

We provide a Hermitian matrix-valued process with correlated diagonal real Brownian motions. Its eigenvalues satisfy a SDE with singular drift of Dyson-type depending on a positive parameter β. The existing correlation is given by $1-(\beta/2)$. For $\beta = 2$, we recover the Hermitian Brownian motion (Dyson model). It is worth-noting that for $\beta \neq 2$, these eigenvalues behave differently from the Dyson eigenvalues : the Vandermonde function, say V is harmonic with respect to the eigenvalues process generator for all β so that no collisions are allowed and a V-transform property holds.

1. Introduction

Let us first introduce the $m \times m$ Hermitian Brownian motion known as the Dyson model ([**48**]) :

$$X_{ij}(t) = \begin{cases} W_{ii}(t) & \text{if} \quad 1 \leq i = j \leq m \\ \left(\dfrac{W_{ij}^1(t) + \sqrt{-1}W_{ij}^2(t)}{\sqrt{2}}\right) & \text{if} \quad 1 \leq i < j \leq m \end{cases}$$

where $(W_{ij})_{i,j}$, $(W_{ij}^1)_{i,j}$, $(W_{ij}^2)_{i,j}$ are independent families of independent standard BMs. If $(r_1, \ldots, r_m)$ denote the eigenvalues of X, then, for all $t \geq 0$

$$dr_i(t) = dW_i(t) + \sum_{i \neq j} \frac{dt}{r_i(t) - r_j(t)}, \quad 1 \leq i \leq m.$$

where $r_1(0) > \ldots r_m(0)$ and $(W_i)_{1 \leq i \leq m}$ is a m-dimensional BM. The matrix above is unitary invariant and its density writes :

$$p(dX) = C_{m,t} e^{-(\mathrm{tr}(X)^2/2t)} \prod_{i \leq j}^{m} d\Re(X_{ij}) \prod_{i<j} d\Im(X_{ij})$$

Hence, the eigenvalues density is :

$$p(r) = C_{m,t} e^{-\sum_{i=1}^m r_i^2/2t} \prod_{i<j} |r_i - r_j|^2 \prod_{i=1}^m dr_i$$

Analogous real symmetric and Hermitian self-dual ([**88**]) models give rise to SDE below for all $t \geq 0$:

$$dr_i(t) = dW_i(t) + \frac{\beta}{2}\sum_{i\neq j}\frac{dt}{r_i(t)-r_j(t)},\ \beta = 1,\,4,\ 1 \leq i \leq m. \tag{38}$$

with respective densities :

$$p(r) = C_{m,t}e^{-\sum_{i=1}^{m} r_i^2/2t}\prod_{i<j}|r_i - r_j|^{\beta}\prod_{i=1}^{m} dr_i, \quad \beta = 1,\,4. \tag{39}$$

Let us consider (38) with arbitrary $\beta > 0$: it was shown in [**28**] that the SDE has a unique strong solution for all $t \geq 0$. The particles model was called in Chapter 4 "β-Dyson". It is quite natural to wonder if (39) do correspond to the density of the process given by (38) in that case. The answer comes from the Dunkl theory. For details, we refer the reader to our previous work on radial Dunkl processes and references there in. In the static regime $t = 1$, the models above are known to be matrices from GOE, GUE and GSE ([**88**]) whose entries are independent Normal variables. Is there an underlying matrix ensemble with independent entries whose eigenvalues density is given at $t = 1$ by (39) with arbitrary Dyson index $\beta > 0$. The answer was already supplied in ([**45**]) with a tridiagonal matrix involving independent Normal and χ distributions. This gave rise to the so-called β-Hermite ensemble. Nevertheless, the method used there fails when dealing with stochastic processes, i. e, with BMs instead of Normal variables. That was at the origin of this work. Our aim was to give a "suitable" model corresponding to "β-Dyson eigenvalues, in the sense that there are as less as possible correlated entries. Unfortunately, we could set a little different Hermitian model with similarly correlated diagonal entries. The correlation is given by $1 - (\beta/2)$. Henceforth, it will be called β-Hermitian model. What is quite interesting is that the correlation does not allow collision between particles while for the β-Dyson model, this depends on β ([**28**]). Indeed, the Vandermonde function, say V is harmonic with respect to the eigenvalues generator. Besides, we write this matrix-valued process as a rank-one diagonal random perturbation of the Dyson-model and we set a V-transform property.

2. The β-Hermitian Brownian Motion.

PROPOSITION 2.1. *Let us consider the Hermitian matrix-valued process $(X_t)_{t\geq 0} = (X_{ij}(t))_{t\geq 0}$ defined by :*

$$X_{ij}(t) = \begin{cases} B_{ii}(t) & \text{if} \quad i = j \\ \sqrt{\dfrac{\beta}{2}}\left(\dfrac{B^1_{ij}(t) + \sqrt{-1}B^2_{ij}(t)}{\sqrt{2}}\right) & \text{if} \quad i > j \end{cases}$$

where $(B_{ii})_{1\leq i\leq m}, (B^1_{ij})_{1\leq j<i\leq m}, (B^2_{ij})_{1\leq j<i\leq m}$ *are three independent families of Brownian motions such that :*

$$d < B_{ii}, B_{kk} >_t = \left(1-\frac{\beta}{2}\right)dt := (1-\rho)dt \qquad 1\leq i\neq k\leq m,$$

while independence is required for the two latters. Then, the eigenvalues process $(\lambda_i(t))_{t\geq 0}$ *of* $(X_t)_{t\geq 0}$ *satisfies :*

$$d\lambda_i(t) = dB_i(t) + \frac{\beta}{2}\sum_{j\neq i}\frac{1}{\lambda_i(t)-\lambda_j(t)}dt, \quad 1\leq i\leq m,\, t<\tau \tag{40}$$

where $\lambda_1(0) > \cdots > \lambda_m(0)$, $\tau := \inf\{t>0,\, \lambda_i(t) = \lambda_j(t), \textit{for some}\,(i,j)\}$ *is the first collision time and* $(B_i)_{1\leq i\leq m}$ *is a family of non-independent Brownian motions. The existing correlation is given by :* $\langle B_i, B_j\rangle_t = (1-\rho)t$*.*

Proof : our strategy relies on Bru's method ([**20**]) summarized in [**74**] : let $(U_t)_{t\geq 0}$ denote the unitary matrix-valued process that diagonalises $(X_t)_{t\geq 0}$. Set :

$$d\Gamma_{ij}(t) = \langle (U^\star(dX)U)_{ij}, (U^\star(dX)U)_{ji}\rangle_t \tag{41}$$

Then

$$d\lambda_i(t) = dM_i(t) + dV_i(t), \quad t<\tau, \quad 1\leq i\leq m,$$

where M_i and V_i denote respectively the local martingale and the finite variation parts :

$$\begin{aligned}\langle dM_i\rangle_t &= d\Gamma_{ii}(t)\\ dV_i(t) &= \sum_{j\neq i}\frac{\Gamma_{ij}(t)}{\lambda_i(t)-\lambda_j(t)}dt + \mathrm{FV}\,(U_t^\star dX_t U_t)_{ii}\end{aligned}$$

The bracket of two entries is given by :

$$\langle dx_{ij}, dx_{kl}\rangle_t = [(1-\rho)\delta_{ij}\delta_{kl} + \rho\delta_{il}\delta_{jk}]t \tag{42}$$

Using $\sum_k \overline{u_{ki}}u_{kj} = \delta_{ij}$, one gets :

$$\begin{aligned}d\Gamma_{ij}(t) &= \sum_{k,l,r,s}\overline{u_{ki}}u_{lj}\overline{u_{rj}}u_{si}d\langle x_{kl}, x_{rs}\rangle_t\\ &= \left[(1-\rho)\sum_{k,r}\overline{u_{ki}}u_{kj}\overline{u_{rj}}u_{ri} + \rho\sum_{k,l}\overline{u_{ki}}u_{ki}\overline{u_{li}}u_{li}\right]dt = [(1-\rho)\delta_{ij}+\rho]\,dt\end{aligned}$$

which yields $\Gamma_{ij}(t)dt = \rho dt$, $i\neq j$ and $\Gamma_{ii}(t)dt = dt$. Hence, since $\mathrm{FV}\,(U_t^\star dX_t U_t) = 0$, we finally get :

$$d\lambda_i(t) = dB_i(t) + \rho\sum_{j\neq i}\frac{1}{\lambda_i(t)-\lambda_j(t)}dt = dB_i(t) + \frac{\beta}{2}\sum_{j\neq i}\frac{1}{\lambda_i(t)-\lambda_j(t)}dt \tag{43}$$

where $(B_i)_{1\leq i\leq m}$ is a family of non-independent Brownian motions. The correlation is given by :

$$\begin{aligned} d\langle B_p, B_q\rangle_t = d\langle \lambda_p, \lambda_q\rangle_t &= \sum_{k,l,r,s} \overline{u_{kp}} u_{lp} \overline{u_{rq}} u_{sq} d\langle x_{kl}, x_{rs}\rangle_t \\ &= \left[(1-\rho)\sum_{k,l} \overline{u_{kp}} u_{kp} \overline{u_{rq}} u_{rq} + \rho \sum_{k,l} \overline{u_{kp}} u_{kq} \overline{u_{lq}} u_{lp}\right] dt = (1-\rho)dt \end{aligned}$$

for $p \neq q$. ■

Remarks. 1/ For $\beta = 2\,(\rho = 1)$, $(X_t)_{t\geq 0}$ is the Hermitian Brownian motion.
2/ The β-Hermitian process can be expressed in terms of the Dyson model : indeed the diagonal vector $\mathbb{B} := (B_{ii})_{1\leq i\leq m}$ can be written as $\mathbb{B} = OW$ where O is a real matrix and $W = (W_{ii})_{1\leq i\leq m}$ is the diagonal of the Dyson model. Moreover,

$$\left.\begin{array}{c} \langle B_{ii}, B_{jj}\rangle_t = (1-\rho)t \\ \langle B_{ii}, B_{ii}\rangle_t = t \end{array}\right\} \Rightarrow \left.\begin{array}{c} (OO^T)_{ij} = (1-\rho) \\ (OO^T)_{ii} = 1 \end{array}\right\} \Rightarrow OO^T = \rho I_m + (1-\rho)J$$

where J is the matrix whose all entries are equal to one. The last equality makes sense if and only if $\rho + m(1-\rho) \geq 0$, that is $\rho \leq m/(m-1)$. In addition, since $J^T = J$ and $J^2 = m\,J$, then $O = \sqrt{\rho} I_m + \theta_0 J$, where θ_0 is a root of $m\theta^2 + 2\sqrt{\rho}\theta - (1-\rho) = 0$. With this decomposition, $X_t = \sqrt{\rho}\,K_t + \theta_0\,\mathrm{tr}(K_t) I_m = \sqrt{\rho}\,K_t + \sqrt{m}\theta_0\Gamma_t I_m$, where K is the Hermitian Dyson model and Γ is a standard Brownian motion ($\sqrt{m}\Gamma_t = \sum_{i=1}^m W_{ii}(t)$). As a result, X is invariant under conjugation by a unitary matrix U. Since K commutes with I_m, then $\lambda_i(t) = \sqrt{\rho} r_i(t) + \theta_0 \sum_{j=1}^m W_{ii}(t)$ where r_i is the i-th eigenvalue of the Dyson Brownian motion satisfying. Note that the W_i's are determined by (41) with $i = j$ and one can easily recover (40). This representation extends even to the case where $\rho + m(1-\rho) < 0$, for θ_0 is a complex number and O is a complex matrix.
Conversely, let us consider $X_t = \sqrt{\rho} K_t + \theta\,\mathrm{tr}(K_t) I_m dt$ for some θ. Then, its eigenvalues satisfy

$$d\lambda_i(t) = \underbrace{\sqrt{\rho} dW_i(t) + \theta \sum_{j=1}^m dW_{ii}(t)}_{dY_i(t)} + \rho \sum_{j\neq i} \frac{dt}{\lambda_i(t) - \lambda_j(t)}$$

One hope finding some value θ_0 of θ such that $(Y_i)_i$ are independent Brownian motions. Nonetheless, this is not possible since on one hand $\langle Y_i\rangle_t = (\rho + m\theta^2 + 2\sqrt{\rho}\theta)t$ and on the other hand $\langle Y_i, Y_j\rangle_t = (m\theta^2 + 2\sqrt{\rho}\theta)t$. We can deepen our line of thinking and look for a one-dimensional real local martingale Z (that may depend on K) such that the eigenvalues of $X_t = \sqrt{\rho} K_t + \theta Z_t I_m dt$ provides a β-Dyson model for some θ. Proceeding as before, one gets :

$$\begin{aligned} \langle \theta Z + \sqrt{\rho} W_i, \theta Z + \sqrt{\rho} W_i\rangle_t &= t \quad 1 \leq i \leq m \\ \langle \theta Z + \sqrt{\rho} W_i, \theta Z + \sqrt{\rho} W_j\rangle_t &= 0 \quad 1 \leq i \neq j \leq m, \end{aligned}$$

which is not possible since by Dubins-Schwarz theorem, $\theta Z = \nu_i - \sqrt{\rho} W_i = \nu_j - \sqrt{\rho} W_j$ for $i \neq j$, where ν_i, ν_j are independent Brownian motions.

2.1. Some properties of the eigenvalues process.

PROPOSITION 2.2. *Let V be the Vandermonde function :*

$$V(x_1, \ldots, x_m) = \prod_{i<j}(x_i - x_j), \qquad x_1 > \ldots > x_m$$

then, the process R defined by : $R_t = (V(\lambda_1(t), \ldots, \lambda_m(t)))^{-1}$ is a continuous local martingale.

Proof : We have to prove that $\mathscr{A}(1/V) = 0$, where $\mathscr{A}$ is given by :

$$\begin{aligned}\mathscr{A} &= \frac{1}{2}\left[\sum_{i=1}^m \partial_i^2 + \beta \sum_{i=1}^m \sum_{j\neq i} \frac{1}{x_i - x_j}\partial_i + \left(1 - \frac{\beta}{2}\right)\sum_{i=1}^m \sum_{j\neq i} \partial_{ij}\right] \\ &= \frac{1}{2}\left[\sum_{i=1}^m \partial_i^2 + \beta \sum_{i=1}^m (\partial_i \log V)\partial_i + \left(1 - \frac{\beta}{2}\right)\sum_{i=1}^m \sum_{j\neq i} \partial_{ij}\right]\end{aligned}$$

On the other hand, using $\partial_i V = V \partial_i \log V$, we have the following derivatives :

$$\begin{aligned}\partial_i(1/V) &= -\frac{\partial_i \log V}{V}, \quad \partial_i \log V = \sum_{k\neq i} \frac{1}{(x_i - x_k)} \\ \partial_i^2(1/V) &= \frac{(\partial_i \log V)^2 - \partial_i^2 \log V}{V}, \qquad \partial_i^2 \log V = -\sum_{k \neq i} \frac{1}{(x_i - x_k)^2} \\ \partial_{ij}(1/V) &= \frac{(\partial_i \log V)(\partial_j \log V) - \partial_{ij} \log V}{V}, \quad \partial_{ij} \log V = \frac{1}{(x_i - x_j)^2}, \; i \neq j.\end{aligned}$$

Hence,

$$\begin{aligned}2h\mathscr{A}(1/V) = (1-\beta)\sum_{i=1}^m (\partial_i \log V)^2 - \sum_{i=1}^m \partial_i^2 \log V + \left(1 - \frac{\beta}{2}\right)\sum_{i=1}^m \sum_{j\neq i} (\partial_i \log V)(\partial_j \log V) \\ - \left(1 - \frac{\beta}{2}\right)\sum_{i=1}^m \sum_{j\neq i} (\partial_i \log V)^2\end{aligned}$$

$$= (1-\beta)\sum_{i=1}^{m}\left(\sum_{k\neq i}\frac{1}{(x_i-x_k)}\right)^2 + \sum_{i=1}^{m}\sum_{k\neq i}\frac{1}{(x_i-x_k)^2} - \left(1-\frac{\beta}{2}\right)\sum_{i=1}^{m}\sum_{j\neq i}\frac{1}{(x_i-x_k)^2}$$
$$+ \left(1-\frac{\beta}{2}\right)\sum_{i=1}^{m}\sum_{j\neq i}\sum_{k\neq i}\sum_{p\neq j}\left(\frac{1}{(x_i-x_k)}\right)\left(\frac{1}{(x_j-x_p)}\right)$$
$$= (1-\beta)\sum_{i=1}^{m}\left(\sum_{k\neq i}\frac{1}{(x_i-x_k)}\right)^2 + \sum_{i=1}^{m}\sum_{k\neq i}\frac{1}{(x_i-x_k)^2} - \left(1-\frac{\beta}{2}\right)\sum_{i=1}^{m}\sum_{j\neq i}\frac{1}{(x_i-x_k)^2}$$
$$+ \left(1-\frac{\beta}{2}\right)\sum_{i,j,k,p\,\text{distinct}}\frac{1}{(x_i-x_k)(x_j-x_p)} - \left(1-\frac{\beta}{2}\right)\sum_{i=1}^{m}\sum_{j\neq i}\frac{1}{(x_i-x_k)^2}$$

As a result

$$\begin{aligned}2V\mathscr{A}(1/V) &= (1-\beta)\sum_{i,j,k\,\text{distinct}}\frac{1}{(x_i-x_j)(x_i-x_k)} + \left(1-\frac{\beta}{2}\right)\sum_{i,j,k,p\,\text{distinct}}\frac{1}{(x_i-x_k)(x_j-x_p)}\\ &=: (1-\beta)U_1 + \left(1-\frac{\beta}{2}\right)U_2\end{aligned}$$

since

$$\sum_{i=1}^{m}\left(\sum_{k\neq i}\frac{1}{(x_i-x_k)}\right)^2 = \sum_{i=1}^{m}\sum_{k\neq i}\frac{1}{(x_i-x_k)^2} + \sum_{i,j,k\,\text{distinct}}\frac{1}{(x_i-x_j)(x_i-x_k)}$$

Finally, it remains to show that $U_1 = U_2 = 0$. Indeed, these functions are invariant under permutations and we can easily see that :

$$\begin{aligned}3U_1 &= \sum_{i,j,k\,\text{distinct}}\left[\frac{1}{(x_i-x_j)(x_i-x_k)} + \frac{1}{(x_k-x_j)(x_k-x_i)} + \frac{1}{(x_j-x_i)(x_j-x_k)}\right] = 0\\ 2U_2 &= \sum_{i,j,k,p\,\text{distinct}}\left[\frac{1}{(x_i-x_k)(x_j-x_p)} + \frac{1}{(x_k-x_i)(x_j-x_p)}\right] = 0\end{aligned}$$

Corollary 2.1. *If $\lambda_1(0) > \cdots > \lambda_m(0)$, then $\tau = \infty$ a. s.*

Proof : This follows from the fact that R is a time changed Brownian motion and can not tend to infinity without infinite oscillations. ■

Lemma 2.1. *Let $(\gamma_1, \dots, \gamma_m)$ be a vector of m correlated Brownian motions such that :*

$$\langle \gamma_i, \gamma_j\rangle_t = \left(1-\frac{\beta}{2}\right)t, \qquad i\neq j$$

Let L denote its infinitesimal generator, then $L(V) = 0$.

Proof : we have :

$$L = \frac{1}{2}\left[\sum_{i=1}^{m}\partial_i^2 + \left(1-\frac{\beta}{2}\right)\sum_{i=1}^{m}\sum_{j\neq i}\partial_{ij}\right]$$

Since $\triangle V = 0$, we have to show that : $\sum_{i=1}^{m}\sum_{j\neq i}\partial_{ij}V = 0$. The same computations as in Proposition 2.2 yield :

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j\neq i}\partial_{ij}V &= V\sum_{i=1}^{m}\sum_{j\neq i}\left[(\partial_i \log V)(\partial_j \log V) + \partial_{ij}\log V\right]\\ &= V\sum_{i=1}^{m}\sum_{j\neq i}\left[-\frac{1}{(x_i-x_j)^2} + \frac{1}{(x_i-x_j)^2}\right] + U_2 = 0 \qquad \blacksquare\end{aligned}$$

REMARK. We can see by a similar computation that, for $\alpha > 0$, $L(V^\alpha) = 0 \Leftrightarrow \alpha = 1$.

PROPOSITION 2.3. *The β-eigenvalues process $(\lambda_1, \dots, \lambda_m)$ is the V-transform of $(\gamma_1, \dots, \gamma_m)$.*

Proof : Let us recall that for a given function harmonic with respect to a generator $\mathscr{L}$ ([**101**]), the generator of the h-process is given by :

$$\mathscr{L}^h := \frac{1}{h}\mathscr{L}h$$

which may take the form

$$\mathscr{L}^h = \mathscr{L} + \Gamma(\cdot, \log h)$$

where $\Gamma(\cdot,\cdot)$ is the so-called " opérateur du carré de champ" defined for any functions f, g in the domain of L by :

$$\Gamma(f,g) = L(fg) - f\,L(g) - g\,L(f)$$

Specializing $h = V$, we shall show that :

$$\Gamma(f, \log V) = \frac{\beta}{2}\sum_{i=1}^{m}\sum_{j\neq i}\frac{1}{(x_i - x_j)}\partial_i f$$

Let $G = \triangle/2$. then, we have :

$$\begin{aligned}\Gamma(f,g) &= G(gf) - f\,G(g) - g\,G(f) + \frac{1-\beta/2}{2}\sum_{i\neq j}(\partial_{ij}(fg) - f\,\partial(g) - g\,\partial(f))\\ &= \tilde{\Gamma}(f,g) + \frac{1-\beta/2}{2}\sum_{i\neq j}(\partial_i f\partial_j g + \partial_j f\partial_i g) = \tilde{\Gamma}(f,g) + \left(1-\frac{\beta}{2}\right)\sum_{i\neq j}\partial_i f\partial_j g\end{aligned}$$

where $\tilde{\Gamma}$ is the " opérateur du carré de champ" associated to G. Since the eigenvalues process of the Dyson model ($\beta = 2$) is the V-transform of an m-dimensional Brownian motion, one has :

$$\tilde{\Gamma}(f, \log V) = \sum_{i \neq j} \frac{1}{x_i - x_j} \partial_i f$$

Furthermore,

$$\begin{aligned}\sum_{i \neq j} \partial_i f \partial_j \log V &= \sum_{i \neq j, k \neq j} \frac{1}{x_j - x_k} \partial_i f = \sum_{i,j,k \,\text{distinct}} \frac{1}{x_j - x_k} \partial_i f - \sum_{i \neq j} \frac{1}{x_i - x_j} \partial_i f \\ &=: U_3 - \sum_{i \neq j} \frac{1}{x_i - x_j} \partial_i f\end{aligned}$$

Finally,

$$2U_3 = \sum_{i,j,k \,\text{distinct}} \left[\frac{1}{x_j - x_k} + \frac{1}{x_k - x_j} \right] \partial_i f = 0$$

which ends the proof. ∎

REMARK. This implies that $\tau = \infty$ almost surely. Indeed, from the first expression of $\mathscr{L}^V$, one deduce that $1/V(\lambda_1, \ldots, \lambda_m)$ is a local martingale.

PROPOSITION 2.4. *A real symmetric matrix-valued process with independent continuous martinagles with the same law whose eigenvalues satisfy*

$$dr_i(t) = dW_i(t) + \frac{1}{2} \sum_{i \neq j} \frac{dt}{r_i(t) - r_j(t)}, \; 1 \leq i \leq m$$

exists if and only if $\beta = 1$.

Proof : The sufficient condition is obvious since for $\beta = 1$, the β-process corresponds to the symmetric Brownian motion. So, let us prove the necessary condition. For symmetric processes, U is orthogonal and $\Gamma_{ij}(t)$ is written :

$$d\Gamma_{ij}(t) = \sum_{k,l,r,s} u_{ki} u_{lj} u_{rj} u_{si} d\langle x_{kl}, x_{rs} \rangle_t$$

and we hope that it equals to

$$[(1 - \rho)\delta_{ij} + \rho] dt$$

The required independence together with the symmetry implies

$$\begin{aligned} d\Gamma_{ij}(t) &= \sum_{k,l} u_{ki} u_{lj} u_{kj} u_{li} d\langle x_{kl}, x_{kl} \rangle_t + \sum_{k,l} u_{ki} u_{lj} u_{lj} u_{ki} d\langle x_{kl}, x_{lk} \rangle_t \\ &= [(U^T U)_{ij}]^2 + (U^T U)_{ii} (U^T U)_{jj}] df(t) \end{aligned}$$

where we set $\langle x_{kl}, x_{kl}\rangle_t = \langle x_{kl}, x_{lk}\rangle_t = f(t)$ for all $1 \leq k, l \leq m$, while all other brackets vanish. Specializing to both cases $i = j$ and $i \neq j$ gives $f(t) = 1/2 = \rho$ which finishes the proof. ■

CHAPITRE 6

Free Jacobi Process

This paper will appear in the Journal of Theoretical Probability.

In this paper, we define and study two parameters dependent free processes (λ, θ) called *free Jacobi*, obtained as the limit of its matrix counterpart when the size of the matrix goes to infinity. The main result we derive is a free SDE analogous to that satisfied in the matrix setting, derived under injectivity assumptions. Once we did, we examine a particular case for which the spectral measure is explicit and does not depend on time (stationary). This allows us to determine easily the parameters range ensuring our injectivity requirements so that our result applies. Then, we show that under an additional condition of invertibility at time $t = 0$, this range extends to the general setting. To proceed, we set a recurrence formula for the moments of the process via free stochastic calculus.

1. Introduction

The classification of classical diffusions relies on three central and interrelated processes : Brownian motion, squared Bessel and Jacobi processes. The two latters can be defined as (see [**101**]) the unique strong solutions of

$$\begin{aligned} dR_t &= 2\sqrt{Z_t}dW_t + \delta\, dt \\ dJ_t &= 2\sqrt{J_t(1-J_t)}dB_t + (p - (p+q)J_t)\, dt \end{aligned}$$

respectively, where δ, p, q are positive and W, B are two standard BMs. Except for the BM, these names are referring to Laguerre and Jacobi polynomials which are eigenfunctions of the corresponding generators (see [**6**], [**114**]). A similar statement holds for BMs with Hermite polynomials. Then, their matrix extensions were developed through several works by Dyson for Hermitian Brownian matrices, Bru ([**19**]) and others for Wishart and Laguerre processes and Doumerc for real and complex matrix Jacobi processes([**43**]). A parallel interpretation using multivariate orthogonal polynomials can be found in [**5**] and [**84**]. Then, it was quite natural to have an insight into the infinite dimensional case, that is when the size of the matrix goes to infinity. This started with Voiculescu for independent large random matrices in the so-called *Gaussian unitary ensemble* ([**111**]). In this way, several results were derived for unitary matrices and in particular unitary processes

([**12**], [**65**]). Few years later, free Wishart processes appeared in [**24**]. They are one parameter-dependent processes defined as a limit of their matrix analogs, Laguerre processes. Authors extend well-known results from matrix theory to this context via free stochastic calculus. For instance, a free SDE of squared Bessel type was derived. All what we said can be summarized in the array drawn below :

matrix size	Hermite	Laguerre	Jacobi
$d = 1$	Br. motion	Squared Bessel	Jacobi
$d > 1$	Hermitian Br. matrix	Wishart/Laguerre	matrix Jacobi
$d = \infty$	Free Br. motion	Free Wishart	?

Our task consists in filling the remaining empty box. Our approach follows the one in [**24**] however, as we will see and as always, the Jacobi setting is more sophisticated and needs more computations. Here, we do recall some definitions and fix some notations that will be frequently used throughout the paper.

2. Definitions and Notations

2.1. Matrix Jacobi process. We refer to [**43**] for facts on real matrix Jacobi processes. In the sequel, we are interested in its complex analog. Let $Y(d)$ be a $d \times d$ unitary Brownian matrix, that is a unitary matrix-valued process such that :

– $Y_0(d) = I_d$

– $(Y_{t_i}(d)Y_{t_{i-1}}^{-1}(d), 1 \leq i \leq n)$ are independent for any collection $0 < t_1 < \ldots < t_n$,

– $Y_t(d)Y_s^{-1}(d)$, $s < t$ has the same distribution as $Y_{t-s}(d)$ ([**12**]).

Let $1 \leq m$, $p \leq d$ and denote by X the $m \times p$ upper left corner of $Y(d)$:

$$X \oplus 0 = P_m Y_t(d) Q_p := \begin{pmatrix} I_m & \\ & 0 \end{pmatrix} Y_t(d) \begin{pmatrix} I_p & \\ & 0 \end{pmatrix}$$

Then $J(m) := XX^\star$ is a $m \times m$ complex matrix Jacobi process of parameters $(p, d-p)$ such that $0 \leq J_t(m) \leq I_m$. If X_t is the $m \times p$ left corner of $\tilde{Z}Y_t(d)$ where $\tilde{Z}$ is a $d \times d$ unitary random matrix independent of Y, then $XX^\star$ is a $m \times m$ complex matrix Jacobi process starting from $X_0X_0^\star$. As for the real matrix case ([**43**]), $I_m - J$ is still a complex matrix Jacobi process of parameters $(d-p, p)$.

2.2. Free probability. Recall that a non commutative probability space (NCPS) is given by a unital algebra $\mathscr{A}$ with a linear functional $\Phi : \mathscr{A} \to \mathbb{C}$. An element in $(\mathscr{A}, \Phi)$ is called a random variable. The subalgebras $(\mathscr{A}_i)_{i \in I}$ are said to be free if for all $a_i \in \mathscr{A}_{j_i}$ such that $\Phi(a_i) = 0$ one has

$$\Phi(\prod_{j \in J} a_i) = 0, \quad j_i \in I, \quad j_i \neq j_{i+1}.$$

$a_1, \dots a_n \in \mathscr{A}$ are free if the subalgebras $\mathscr{A}$ generated by $\{\mathbf{1}, a_i\}$ are free ($\mathbf{1}$ denotes the unit of $\mathscr{A}$). The distribution of a random variable $a \in \mathscr{A}$ is given by its moments $\Phi(a^r)$, $r \geq 0$. Similarly, the distribution of $a_1, \dots, a_n$ is given by $\Phi(L(a_1, \dots, a_n))$ for all non commutative polynomial $L \in \mathbb{C}[a_1, \dots, a_n]$. When this family is free, this factorizes into products of moments of a_i so that it is entirely determined by a_i's distributions. A famous realization of random variables is illustrated by random matrices of all order finite moments : the algebra is

$$\mathscr{A}_d := \bigcap_{p>0} L^p\left(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}\right) \otimes \mathscr{M}_d(\mathbb{C})$$

where $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ is a filtered probability space and $\mathscr{M}_d(\mathbb{C})$ stands for the set of $d \times d$ complex matrices, equipped with the normalized trace expectation $\mathbb{E} \otimes \mathrm{tr}_d$. We say that the family of $d \times d$ random matrices $(A_s(d))_{s \in S}$ converge in distribution to the family of random variables $(a_s)_{s \in S}$ in some NCPS $(\mathscr{A}, \Phi)$ if and only if :

$$\lim_{d \to \infty} \mathbb{E}[\mathrm{tr}_d(A_{s_1}(d) \dots A_{s_r}(d))] = \Phi(a_{s_1} \cdots a_{s_r}), \quad s_1, \cdots s_r \in S.$$

which implies that

$$\lim_{d \to \infty} \mathbb{E}[\mathrm{tr}_d(A_s^k(d)] = \Phi(a_s^k), \quad k \geq 1.$$

$(A_s(d),\, s \in S)$ is said to be *asymptotically free* if $(a_s)_{s \in S}$ form a free family. As stated before, independent random matrices enjoying some invariance properties are shown to be asymptotically free random variables in some NCPS. The starting point was with Voiculescu for independent $d \times d$ matrices belonging to the GUE with variance $1/d$ ([**111**]). This is used to show that the normalized Hermitian BM converges in distribution to the *free additive Brownian motion* : it is a collection of self-adjoint random variables indexed by time, say $(a_t)_{t \geq 0}$ (or process) with free increments $(a_t - a_s,\, s < t)$ and such that $a_t - a_s$ has the same law as a_{t-s} given by the semicircle law σ_{t-s} (free additive Lévy process, [**24**]) where

$$\sigma_t(dx) = \frac{1}{2\pi t}\sqrt{4t - x^2}\mathbf{1}_{[-2\sqrt{t}, 2\sqrt{t}]}(x)dx.$$

A similar result we will use later is due to Biane ([**12**]) : the unitary Brownian matrix converges in distribution to the free multiplicative Brownian motion Y in some NCPS $(\mathscr{A}, \Phi)$. Recall that Y is unitary, $Y_0 = \mathbf{1}$, has free left increments, that is, for a collection of times $0 < t_1 < t_2 < \dots < t_n$, $Y_{t_n}Y_{t_{n-1}}^{-1}, \dots, Y_{t_2}Y_{t_1}^{-1}$ are free and the law of Y_t, say ν_t, is given by the so-called Σ-transform (see [**12**]) :

$$\Sigma_{\nu_t}(z) = e^{\frac{t}{2}\frac{1+z}{1-z}}, \qquad \nu_{t+s} = \nu_t \boxtimes \nu_s$$

where $\boxtimes$ denotes the free multiplicative convolution (free multiplicative Lévy process, see [**11**], [**12**]). For our purposes, we shall consider a von Neumann algebra $\mathscr{A}$ endowed with a faithful tracial state Φ (see [**38**] for details). This is known as a $W^\star$ NCPS. The L^q-norm is given by $||a||_{L^q} := \Phi[(aa^\star)^{q/2}]^{1/q}$ for $1 \leq q < \infty$. The

L^∞-norm or the algebra-norm is defined as the limit of the L^q-norm as q tends to infinity. It will be denoted by $||\cdot||_{L^\infty}$ or by $||\cdot||$ if there is no confusion.

3. Free Jacobi Process

Let $Y(d(m))$ be a $d(m)\times d(m)$ unitary Brownian matrix with $m\times p(m)$ upper left corner X such that :

$$\lim_{m\to\infty}\frac{m}{p(m)} = \lambda > 0, \lim_{m\to\infty}\frac{p(m)}{d(m)} = \theta \in]0,1] \quad \text{so that} \quad \lim_{m\to\infty}\frac{m}{d(m)} = \lambda\theta.$$

Let $Q_m := Q_{p(m)}$ with Q_p defined in subsection 2.1. Then, $J_t(m) = X_tX_t^\star$ and :

$$A_t(m) := J_t(m)\oplus 0_{d(m)-m} = P_mY_t(d(m))Q_mY_t^\star(d(m))P_m$$

It follows that :

$$\begin{aligned}
\lim_{m\to\infty}\mathrm{tr}_{d(m)}(P_m) &= \lambda\theta \\
\lim_{m\to\infty}\mathrm{tr}_{d(m)}(Q_m) &= \theta, \quad \theta\in]0,1] \\
\lim_{m\to\infty}\mathbb{E}\left\{\mathrm{tr}_m[J_{t_1}(m)J_{t_2}(m)\dots J_{t_n}(m)]\right\} &= \lim_{m\to\infty}\frac{d(m)}{m}\mathbb{E}\left\{\mathrm{tr}_{d(m)}[A_{t_1}(m)A_{t_2}(m)\dots A_{t_n}(m)]\right\}
\end{aligned}$$

for any collection $t_1,\dots,t_n$. Next, we make use of the following result ([**65**], p. 157) :

THEOREM 3.1. *Let $(U_s(m))_s$ be a family of independent $m\times m$ unitary random matrices such that the distribution of $U_s(m)$ is equal to that of $VU_s(m)V^\star$ for any unitary matrix V (unitary invariant) and such that $U_s(m)$ converges in distribution. Let $(D_t(m))_t$ be a family of $m\times m$ constant matrices converging in distribution and such that $\sup_m||D_t(m)||<\infty$. Then the families*

$$\{U_s(m),U_s^\star(m)\}_s, \{D_t(m),D_t^\star(m),t\geq 0\}$$

are asymptotically free as $m\to\infty$.

Note that $\{Y_t(d(m))Y_s^{-1}(d(m))\}_{0\leq s<t}$ is a unitary invariant family since $(Y_t)_{t\geq 0}$ is right-left invariant. By the freeness of increments of Y mentionned above, Theorem 3.1 claims that :

$$\{(Y_t(d(m)))_{t\geq 0},(Y_t^\star(d(m)))_{t\geq 0}\},\{P_m,Q_m\}$$

are asymptotically free. Thus, its limiting distribution in $(\mathscr{A}_{d(m)},\mathbb{E}\otimes\mathrm{tr}_{d(m)})$ as m goes to infinity is the distribution of $\{(Y_t)_{t\geq 0},(Y_t^\star)_{t\geq 0}\}$, $\{P,Q\}$ in $(\mathscr{A},\Phi)$ such that

- Y is a free multiplicative Brownian motion in $(\mathscr{A},\Phi)$.
- P is a projection with $\Phi(P)=\lambda\theta\leq 1, \quad \theta\in]0,1]$.
- Q is a projection with $\Phi(Q)=\theta$.
- $QP = PQ = \begin{cases} P & \text{if} \quad \lambda\leq 1 \\ Q & \text{if} \quad \lambda > 1\end{cases}$
- $\{(Y_t)_{t\geq 0},(Y_t^\star)_{t\geq 0}\}$ and $\{P,Q\}$ are free.

Hence, we deduce that the limiting distribution of the complex matrix Jacobi process $(J_t)_{t\geq 0}$ in $(\mathscr{A}_m, \mathbb{E}\otimes \mathrm{tr}_m)$ is the distribution of $(PY_tQY_t^\star P)_{t\geq 0}$ in $P\mathscr{A}P$ equipped with the state

$$\widetilde{\Phi} = \frac{1}{\Phi(P)}\Phi_{|P\mathscr{A}P} = \frac{1}{\lambda\theta}\Phi_{|P\mathscr{A}P}.$$

$(P\mathscr{A}P, \widetilde{\Phi})$ is called the *compressed* NCPS. This suggests to define the free Jacobi process as follows :

DEFINITION. Let $(\mathscr{A}, \Phi)$ be a $W^\star$ NCPS. Let $\theta \in]0,1]$ and $\lambda > 0$ such that $\lambda\theta \leq 1$. Let P and Q be two projections such that

$$\Phi(Q) = \theta, \quad \Phi(P) = \lambda\theta, \text{and}\, PQ = QP = \left\{ \begin{array}{lll} P & \text{if} & \lambda \leq 1 \\ Q & \text{if} & \lambda > 1 \end{array} \right.$$

Let Y be a free multiplicative Brownian motion such that $\{(Y_t)_{t\geq 0}, (Y_t^\star)_{t\geq 0}\}$ and $\{P, Q\}$ is a free family in $(\mathscr{A}, \Phi)$. We will say that a process J in a $W^\star$ NCPS (B, Ψ) is a free Jacobi process with parameters (λ, θ), denoted by $FJP(\lambda, \theta)$, if its distribution in (B, Ψ) is equal to the distribution of the process $(PY_tQY_t^\star P)_{t\geq 0}$ in $(P\mathscr{A}P, (1/\Phi(P))\Phi_{|P\mathscr{A}P})$. This process starts from $J_0 = P$ if $\lambda \leq 1$ and $J_0 = Q$ if $\lambda > 1$.
Equivalently, the law of J is the limiting distribution of a complex matrix Jacobi process when $\dfrac{m}{p(m)} \underset{m\to\infty}{\longrightarrow} \lambda$ and $\dfrac{m}{d(m)} \underset{m\to\infty}{\longrightarrow} \lambda\theta$.

We also define the free Jacobi process starting from J_0 :

DEFINITION. Let Y be a free multiplicative Brownian motion and Z a unitary operator free with Y. Then, the process defined by $\tilde{Y} = YZ$ is a free multiplicative Brownian motion starting at $\tilde{Y}_0 = Z$. Moreover, if Z is free with $\{P, Q\}$, then the process $\tilde{J}$ defined by :

$$\tilde{J}_t := P\tilde{Y}_tQ\tilde{Y}_t^\star P$$

is called a free Jacobi process with parameters (λ, θ) and starting from $\tilde{J}_0 = PZQZ^\star P$.

Since $P - J = PY_t(\mathbf{1}-Q)Y_t^\star P$ and $\mathbf{1} - Q$ is a projection, then :

COROLLARY 3.1. *If J is a $FJP(\lambda, \theta)$ with λ, θ as above and starting from J_0, then $P - J$ is still a $FJP(\lambda\theta/(1-\theta), 1-\theta)$ starting from $P - J_0$.*

For the sake of simplicity, we will write Y for a free multiplicative Brownian motion starting from Y_0 and J for a free Jacobi process $(FJP(\lambda, \theta))$ starting at J_0 .

4. Free Jacobi Process And Free Stochastic Calculus

We refer to [**12**] and [**13**] for free stochastic calculus and notations. Let $(\mathscr{A}_t)_{t\geq 0}$ be an increasing family of unital, weakly closed $\star$-subalgebras of the von Neumann algebra $\mathscr{A}$. Then, $(\mathscr{A}, (\mathscr{A}_t), \Phi)$ is called a filtered $W^\star$ NCPS. Since Φ is tracial, there exists a unique conditional expectation denoted by $\Phi(\cdot|\mathscr{A}_t)$. Let $\mathscr{A} \otimes \mathscr{A}^{op}$ be the von Neumann tensor product algebra equipped with the tracial state $\Phi \otimes \Phi^{op}$. A bi-process U is an element in $\mathscr{A} \otimes \mathscr{A}^{op}$ and is written as $(U_t = \sum_i A_t^i \otimes B_t^i)$. It is adapted if $U_t \in \mathscr{A}_t \otimes \mathscr{A}_t$ for all $t \geq 0$. The prefix "bi" and the superscript "op" refer to the fact that the integrator can be multiplied both to the left and to the right due to the non-commutativity. Furthermore, adapted bi-processes form a complex vector space that we endow with the norm :

$$||U||_\infty = \left(\int_0^\infty ||U_s||^2_{L^\infty(\mathscr{A}\otimes\mathscr{A}^{op})} ds\right)^{1/2}$$

where the tensor algebra norm defined by :

$$||\cdot||_{L^\infty(\mathscr{A}\otimes\mathscr{A}^{op})} := \lim_{p\to\infty} ||\cdot||_{L^p(\mathscr{A}\otimes\mathscr{A}^{op})}$$

The completion of this space is denoted by $\mathscr{B}^a_\infty$ and $||\cdot||_{\mathscr{B}^a_\infty}$ will be denoted by $||\cdot||_\infty$. Recall also that, for fixed $t > 0$ and $U \in \mathscr{B}^a_\infty$, we have :

$$\int_0^t U_s \sharp dX_s = \int_0^\infty U_s \mathbf{1}_{[0,t]}(s) \sharp dX_s$$

where X is a free additive Brownian motion and

$$U_s \sharp dX_s := \sum_i A_s^i dX_s B_s^i$$

For any two adapted bi-processes N and M belonging to $\mathscr{B}^a_\infty$:

$$\Phi\left\{\int_0^t N_s \sharp dX_s \int_0^t M_s \sharp dX_s\right\} = \int_0^t < N_s, M_s^\star > ds, \tag{44}$$

where $<>$ is the inner product in $L^2(\mathscr{A}, \Phi) \otimes L^2(\mathscr{A}, \Phi)$ (namely, if $N = a \otimes a'$ and $M = b \otimes b'$, then $M^\star = (b')^\star \otimes b^\star$ and $< N, M^\star >= \Phi(ab')\Phi(a'b)$). Consider the process $(J_t := PY_tQY_t^\star P)_{t\geq 0}$, where P, Q are two projections as in the definition above, Y is a free multiplicative Brownian motion in $(\mathscr{A}, \Phi)$. Recall that Y satisfies the free SDE (see [**12**]) :

$$dY_t = i\, dX_t\, Y_t - \frac{1}{2} Y_t dt, \qquad Y_0 \in \mathscr{A}$$

where $(X_t)_{t\geq 0}$ is a free additive Brownian motion in $(\mathscr{A}, \Phi)$. By free Itô's formula ([**12**], [**13**], [**79**]), we get :

$$\begin{aligned} d(Y_tQY_t^\star) &= (dY_t)QY_t^\star + Y_tQ(dY_t^\star) + \Phi(Y_tQY_t^\star)dt \\ &= idX_tY_tQY_t^\star - iY_tQY_t^\star dX_t - Y_tQY_t^\star dt + \theta dt \end{aligned}$$

since X_t is self-adjoint. Thus, the free Jacobi process satisfies :

$$\begin{aligned} dJ_t &= Pd(Y_t\,Q\,Y_t^\star)P = iPdX_tY_tQY_t^\star P - iPY_tQY_t^\star dX_tP - J_t\,dt + \theta Pdt \\ &= iPY_t(Y_t^\star dX_tY_t)QY_t^\star P - iPY_tQ(Y_t^\star dX_tY_t)Y_t^\star P - J_t\,dt + \theta Pdt, \end{aligned} \tag{45}$$

since Φ is tracial and Y_t is unitary (by definition). The next step consists of characterizing the process $(Y_t^\star dX_tY_t)_{t\geq 0}$. This needs the following characterization of the free additive Brownian motion ([**14**], [**24**]) which is the free analogue of the Lévy characterization :

THEOREM 4.1. *Let $(\mathscr{A}_s,\, s\in[0,1])$ be an increasing family of von Neumann subalgebras in a non-commutative probability space $(\mathscr{A},\Phi)$, and let $(Z_s=(Z_s^1,\dots,Z_s^m);\, s\in[0,1])$ be an m-tuple of self-adjoint $(\mathscr{A}_s)$-adapted processes such that :*
– *Z is bounded and $Z_0=0$.*
– *$\Phi(Z_t^i|\mathscr{A}_s)=Z_s^i$ for all $1\leq i\leq m$.*
– *$\Phi(|Z_t^i-Z_s^i|^4)\leq K(t-s)^2$ for some constant K and for all $1\leq i\leq m$.*
– *For any $l,p\in\{1,\dots,m\}$ and all $A,B\in\mathscr{A}_s$, one has :*

$$\Phi(A(Z_t^p-Z_s^p)B(Z_t^l-Z_s^l)) = \mathbf{1}_{\{p=l\}}\Phi(A)\Phi(B)(t-s)+o(t-s),$$

then Z is a m-dimensional free Brownian motion.

It follows that :

LEMMA 4.1. *The process $(S_t) := (\int_0^t Y_s^\star dX_sY_s)_{t\geq 0}$ is an $\mathscr{A}_t$ - free Brownian motion.*

Proof : one has to check the four conditions mentionned above are satisfied. Note that for all $T>0$, $Y_t^\star\otimes Y_t\mathbf{1}_{[0,T]}\in\mathscr{B}_\infty^a$ since $||Y_t||=||Y_t^\star||=1$. Take $A,\,B\in\mathscr{A}$, then (using (44) in the second line) :

$$\begin{aligned} \Phi(A(S_t-S_s)B(S_t-S_s)) &= \Phi\left\{\int_s^t AY_r^\star dX_rY_r\int_s^t BY_r^\star dX_rY_r\right\} \\ &= \Phi\left\{\int_s^t (AY_r^\star\otimes Y_r)\sharp dX_r\int_s^t (BY_r^\star\otimes Y_r)\sharp dX_r\right\} \\ &= \int_s^t \Phi(Y_rAY_r^\star)\Phi(Y_rBY_r^\star)dr \\ &= \Phi(A)\Phi(B)(t-s), \end{aligned}$$

since Φ is tracial and Y_t is unitary. Hence, the fourth condition is fullfilled. For the third, we follow in the same way and use again the fact that Y_t is unitary to get :

$$\Phi(|S_t-S_s|^4)\leq (t-s)^2,$$

The second condition results from the fact $\left(\int_0^t Y_s^\star\,dX_sY_s\right)_{t\geq 0}$ defines an $\mathscr{A}_t$ - martingale. Finally, it is easily seen from the end of the proof of Theorem 3.2.1 in [**13**] that :

$$||\int_0^t Y_s^\star\, dX_s Y_s||^2 := ||\int_0^t Y_s^\star\, dX_s Y_s||^2_{L^\infty(\mathscr{A})} \leq 8\int_0^t ||Y_s^\star \otimes Y_s||^2_{L^\infty(\mathscr{A}\otimes\mathscr{A}^{op})} ds = 8t$$

since the integrand's norm is equal to 1 from the unitarity of Y_s. ∎

Thus, (45) transforms to :

$$\begin{aligned} dJ_t &= iPY_t\, dS_t\, QY_t^\star P - iPY_tQ\, dS_t\, Y_t^\star P - J_t\, dt + \theta P dt \\ &= iPY_t(1-Q)\, dS_t\, QY_t^\star P - iPY_tQ\, dS_t\,(1-Q)Y_t^\star P - J_t\, dt + \theta P dt \end{aligned}$$

In order to use the polar decomposition of $P - J_t$, we write :

$$P - J_t = (PY_t - PY_tQ)(Y_t^\star P - QY_t^\star P) := C^\star C$$

since Y_t is unitary, $Q^2 = Q$ and $P^2 = P$. Hence

$$QY_t^\star P = R_t\sqrt{J_t} \quad C = (1-Q)Y_t^\star P = V_t\sqrt{P - J_t},$$

which gives :

$$dJ_t = \sqrt{P-J_t}(iV_t^\star\, dS_t\, R_t)\sqrt{J_t} + \sqrt{J_t}((iR_t)^\star\, dS_t\, V_t)\,\sqrt{P-J_t} + (\theta P - J_t)\, dt \tag{46}$$

Remark. An elemantary and needed relation is :

$$\sqrt{J_t}R_t^\star V_t\sqrt{P-J_t} = PY_tQ(1-Q)Y_t^\star P = 0 \tag{47}$$

since $Q = Q^2$.

Proposition 4.1. *Suppose that J_t and $P - J_t$ are injective operators in $P\mathscr{A}P$. Then, the following holds :*

- $R_tP = R_t$ *and* $V_tP = V_t$.
- $R_t^\star R_t = P$ *and* $V_t^\star V_t = P$.
- $PR_t^\star V_tP = PV_t^\star R_tP = 0$.

Proof : Recall first that if T is an operator in $\mathscr{A}$, then the support E of T is the orthogonal projection on $(ker\, T)^\perp = \overline{Im\, T^\star}$ and satisfies $TE = T$ (see A. III in [**38**]). Furthermore, if we consider the polar decomposition of T, namely $T = A|T| = A(T^\star T)^{1/2}$, then E is also the support of A and the latter is partially isometric, that is $A^\star A = E$ and $AA^\star = F$ where F is the support of $T^\star$. Thus, the two first assertions follow if we prove that P is the support of both J_t and $P - J_t$. Indeed, the injectivity of J_t in $P\mathscr{A}P$ implies that $ker\, J_t = ker\, P$. Thus, we claim that P is the support of J_t ($(ker\, P)^\perp = Im\, P$) and the same result holds for $P - J_t$. The third is obvious when J_t and $P - J_t$ are invertible. Else, (47) is written in $P\mathscr{A}P$:

$$0 = \sqrt{J_t}R_t^\star V_t\sqrt{P-J_t} = \sqrt{J_t}(PR_t^\star V_tP)\sqrt{P-J_t}$$

Since both J_t and $P - J_t$ are injective operators in $P\mathscr{A}P$, then :

$$(PR_t^\star V_tP)\sqrt{P-J_t} = 0 \Rightarrow \sqrt{P-J_t}(PV_t^\star R_tP) = 0 \Rightarrow (PV_t^\star R_tP) = 0$$ ∎

Corollary 4.1. *Under the same assumption of Proposition 4.1 , the process* $(W_t)_{t\geq 0}$ *defined by* $W_t := (i/\sqrt{\Phi(P)})\int_0^t (PV_s^\star \otimes R_sP)\sharp dS_s$ *is a* $P\mathscr{A}_tP$ *-complex free Brownian motion.*

Proof : Let us first recall that a process $Z : \mathbb{R}_+ \to \mathscr{A}$ is a complex $(\mathscr{A}_t)$ -Brownian motion if it can be written $Z = (X^1 + \sqrt{-1}X^2)/\sqrt{2}$, where (X^1, X^2) is a 2-dimensional $(\mathscr{A}_t)$ -free Brownian motion. Note also that $(iZ_t)_{t\geq 0}$ is still a complex $(\mathscr{A}_t)$ -Brownian motion since $(-X^2)$ is an $(\mathscr{A}_t)$ -free Brownian motion. So, we shall show that :

$$\left(\tilde{W}_t := (1/\sqrt{\Phi(P)})\int_0^t (PV_s^\star \otimes R_sP)\sharp dS_s\right)_{t\geq 0}$$

is a $P\mathscr{A}_tP$ -complex free Brownian motion. To proceed, it suffices to show that :

$$\begin{aligned}
X_t^1 &= \frac{\tilde{W}_t + \tilde{W}_t^\star}{\sqrt{2}} = \frac{1}{\sqrt{2\Phi(P)}}\left(\int_0^t (PV_s^\star \otimes R_sP)\sharp dS_s + \int_0^t (PR_s^\star \otimes V_sP)\sharp dS_s\right) \\
X_t^2 &= \frac{\tilde{W}_t - \tilde{W}_t^\star}{\sqrt{2}i} = \frac{1}{\sqrt{2\Phi(P)}i}\left(\int_0^t (PV_s^\star \otimes R_sP)\sharp dS_s - \int_0^t (PR_s^\star \otimes V_sP)\sharp dS_s\right)
\end{aligned}$$

define two free $(\mathscr{A}_t)$ -free Brownian motions using again the characterization given in Theorem 4.1. We will do this for X_1. Note that, since R_t and V_t are partially isometric, then $(PV_t^\star \otimes R_tP\mathbf{1}_{[0,T]})_{t\geq 0}$ and $(PR_t^\star \otimes V_tP\mathbf{1}_{[0,T]})_{t\geq 0} \in B_\infty^a\, \forall T > 0$. Hence, the first condition follows since $\left(\int_0^t (PV_s^\star \otimes R_sP)dS_s\right)_{t\geq 0}$ and $\left(\int_0^t (PR_s^\star \otimes V_sP)dS_s\right)_{t\geq 0}$ are $P\mathscr{A}_tP$-martingales. For $A, B \in \mathscr{A}_s$ and using (44), one has :

$$\begin{aligned}
&\tilde{\Phi}(PAP(X_t^1 - X_s^1)PBP(X_t^1 - X_s^1)) = \\
&\frac{1}{2\Phi(P)}\tilde{\Phi}\left(\int_s^t (PAPV_u^\star \otimes R_uP)\sharp dS_u + \int_s^t (PAPR_u^\star \otimes V_uP)\sharp dS_u\right) \\
&+\left(\int_s^t (PBPV_u^\star \otimes R_uP)\sharp dS_u + \int_s^t (PBPR_u^\star \otimes V_uP)\sharp dS_u\right) \\
&= \frac{1}{2\Phi^2(P)}\Phi\left(\int_s^t (PAPV_u^\star \otimes R_uP)\sharp dS_u + \int_s^t (PAPR_u^\star \otimes V_uP)\sharp dS_u\right) \\
&+\left(\int_s^t (PBPV_u^\star \otimes R_uP)\sharp dS_u + \int_s^t (PBPR_u^\star \otimes V_uP)\sharp dS_u\right) \\
&= \frac{1}{2}\int_s^t [\tilde{\Phi}(PAPV_u^\star V_uP)\tilde{\Phi}(R_uPBPR_u^\star) + \tilde{\Phi}(PAPR_u^\star R_uP)\tilde{\Phi}(V_uPBPV_u^\star)]du \\
&+\frac{1}{2}\int_s^t [\tilde{\Phi}(PAPV_u^\star R_uP)\tilde{\Phi}(R_uPBPV_u^\star) + \tilde{\Phi}(PAPR_u^\star V_uP)\tilde{\Phi}(V_uPBPR_u^\star)]du
\end{aligned}$$

$$= \frac{1}{2}\int_s^t [\tilde{\Phi}(PAP)\tilde{\Phi}(PBP) + \tilde{\Phi}(PAP)\tilde{\Phi}(PBP)]du +$$
$$\frac{1}{2}\int_s^t [\tilde{\Phi}(APV_u^\star R_u P)\tilde{\Phi}(BPV_u^\star R_u P) + \tilde{\Phi}(APR_u^\star V_u P)\tilde{\Phi}(BPR_u^\star V_u P)]du$$
$$= \tilde{\Phi}(PAP)\tilde{\Phi}(PBP)(t-s),$$

since $PV_u^\star R_u P = PR_u^\star V_u P = 0$ and since R_u and V_u are partially isometric (Proposition 4.1). Similarly, the same result holds for X^2. Furthermore, one has :

$$\tilde{\Phi}(PAP(X_t^1 - X_s^1)PBP(X_t^2 - X_s^2)) =$$
$$\frac{1}{2\Phi(P)}\tilde{\Phi}\left(\int_s^t (PAPV_u^\star \otimes R_u P)\sharp dS_u + \int_s^t (PAPR_u^\star \otimes V_u P)\sharp dS_u\right)$$
$$+ \left(\int_s^t (PBPV_u^\star \otimes R_u P)\sharp dS_u - \int_s^t (PBPR_u^\star \otimes V_u P)\sharp dS_u\right)$$
$$= \frac{1}{2}\int_s^t [-\tilde{\Phi}(PAPV_u^\star V_u P)\tilde{\Phi}(R_u PBPR_u^\star) + \tilde{\Phi}(PAPR_u^\star R_u P)\tilde{\Phi}(V_u PBPV_u^\star)]du$$
$$+ \frac{1}{2}\int_s^t [\tilde{\Phi}(PAPV_u^\star R_u P)\tilde{\Phi}(R_u PBPV_u^\star) - \tilde{\Phi}(PAPR_u^\star V_u P)\tilde{\Phi}(V_u PBPR_u^\star)]du$$
$$= \frac{1}{2}\int_s^t [-\tilde{\Phi}(PAP)\tilde{\Phi}(PBP) + \tilde{\Phi}(PAP)\tilde{\Phi}(PBP)]du$$
$$+ \frac{1}{2}\int_s^t [\tilde{\Phi}(APV_u^\star R_u P)\tilde{\Phi}(BPV_u^\star R_u P) - \tilde{\Phi}(APR_u^\star V_u P)\tilde{\Phi}(BPR_u^\star V_u P)]du = 0$$

which finishes the proof. Substituting R_t and V_t by R_tP and V_tP in (45) and using Corollary 4.1, we proved :

THEOREM 4.2. *Given J_0 such that J_0 and $P - J_0$ are injective operators in $P\mathscr{A}P$, let $T := \inf\{s, ker(J_s) \neq kerP \text{ or } ker(P - J_s) \neq kerP\} > 0$ by continuity of the trajectories. Then, for all $t < T$,*

$$(48) \qquad \begin{cases} dJ_t &= \sqrt{\lambda\theta}\sqrt{P - J_t}dW_t\sqrt{J_t} + \sqrt{\lambda\theta}\sqrt{J_t}dW_t^\star\sqrt{P - J_t} + (\theta P - J_t)\,dt \\ J_0 &= PY_0QY_0^\star P \end{cases}$$

where $(W_t)_{t\geq 0}$ is a $P\mathscr{A}P$-complex free Brownian motion.

In the remainder of this paper, we will try to find the range of (λ, θ) ensuring the injectivity of both J_t and $P - J_t$. This is equivalent to find (λ, θ) for which the spectral measure of both J_t and $P - J_t$ has no atoms in 0. We first investigate the stationary case then deal with the general setting.

5. Free Jacobi process : the stationary case

In this section, we will give some interest in the particular case when Y_0 is Haar distributed, that is $\Phi(Y_0^k) = \delta_{k0}$. Then Y_t remains Haar distributed for all

$t > 0$. Thus, the law of J_t does not depend on time and such a process is called a stationary free Jacobi process. Its law has already been computed by both Capitaine and Casalis using the so-called *generalized free cumulants* ([**24**]) and Collins ($P = Q$, [**34**]). Here we will use Nica and Speicher's result on compression by free projections. More precisely, authors considered PaP for any operator $a \in \mathscr{A}$ free with P (cf [**90**], [**107**]). This condition is fulfilled for $a = Y_tQY_t^\star$ since Y_t is Haar unitary. In fact, the following classical result holds (see [**65**]) :

LEMMA 5.1. *If U is Haar unitary and $\mathscr{B}$ is a sub-algebra which is free with U, then, $\forall A, B \in \mathscr{B}$, A and $UBU^\star$ are free.*

From [**107**], the law of J_t in $(P\mathscr{A}P, \tilde{\Phi})$ writes :

$$\mu_{J_t} = \boxplus^r \mu_{\lambda\theta a}$$

where $\Phi(P) = \lambda\theta = 1/r$ and $\boxplus$ denotes the free additive convolution. Since Φ is tracial and Q is a projection, then $\Phi(a^k) = \Phi(Q) = \theta$ for all $k \geq 1$. Thus,

$$\mu_a = (1-\theta)\delta_0 + \theta\delta_1$$

Furthermore, the Cauchy transform of a writes

$$G_a(z) := \frac{1}{z} + \sum_{k\geq 1} \frac{\Phi(a^k)}{z^{k+1}} = \frac{z+\theta-1}{z(z-1)},$$

Its inverse is then written :

$$K_a(z) = \frac{z+1+\sqrt{(z-1)^2+4\theta z}}{2z},$$

and finally,

$$R_a(z) := K_a(z) - \frac{1}{z} = \frac{z-1+\sqrt{(z-1)^2+4\theta z}}{2z}.$$

R is known as the R-transform. It plays the role of the log-Laplace transform in classical probability since it linearizes the free additive convolution. This means that if a and b are free, then $R_{a+b} = R_a + R_b$. Hence

$$R_{J_t}(z) = rR_{\lambda\theta a}(z) = R_a(\lambda\theta z)$$

where the last equality follows from the expression of the R-transform in terms of free cumulants and the multilinearity of these latters. It follows that

$$R_{J_t}(z) = \frac{\lambda\theta z - 1 + \sqrt{(\lambda\theta z-1)^2 + 4\lambda\theta^2 z}}{2\alpha z} = \frac{z-r+\sqrt{(z-r)^2+4z/\lambda}}{2z},$$

which implies that :

$$K_{J_t}(z) = R_{J_t} + \frac{1}{z} = \frac{z+(2-r)+\sqrt{(z-r)^2+4z/\lambda}}{2z}.$$

which inverse is :

$$G_{J_t}(z) = \frac{(2-r)z + (1/\lambda - 1) + \sqrt{Az^2 - Bz + C}}{2z(z-1)},$$

where $A = r^2 = 1/(\lambda\theta)^2$, $B = 2(r + (r-2)/\lambda)$ et $C = (1 - 1/\lambda)^2$. Since J_t is selfadjoint and $0 \leq J_t \leq P$ then its spectrum lies in $[0,1]$. Thus $z \in \mathbb{C} \setminus [0,1]$ and is constrained to $\Im[G(z)] < 0$ when $\Im(z) > 0$ which determines the square root. The law of J_t takes the form :

$$\mu_{J_t}(dx) = a_0\delta_0(dx) + a_1\delta_1(dx) + g(x)dx,$$

where

$$\begin{aligned} a_0 &= \lim_{y\to 0^+} -y\Im[G(iy)], \quad a_1 = \lim_{y\to 0^+} -y\Im[G(1+iy)] \\ g(x) &= \lim_{y\to 0^+} -\frac{1}{\pi}\Im[G(x+iy)] \qquad \text{for some } x \in (0,1), \end{aligned}$$

Remark. The last equality holds whenever $\lim_{z\in D\to x} \Im(G(z)) = \Im(G(x))$ where D is the upper half-plane and $x \in \mathbb{R}$ (cf [**32**]). In fact, from

$$G_F(z) = \int \frac{dF(\zeta)}{z - \zeta}$$

for some distribution function F, the following inversion formula holds (cf [**65**], [**107**]) :

$$F(b) - F(a) = \lim_{y\to 0^+} -\frac{1}{\pi}\int_a^b \Im(G(x+iy))dx$$

for any two continuity points a, b of F (weak convergence). Silverstein and Choi showed that if the limit above exists, then F is differentiable and dF has the density function with respect to the Lebesgue measure given by $F'(x) = lim_{y\to 0^+} - (1/\pi)\Im(G(x+iy))$. See [**32**] for more details.

Proposition 5.1. $a_0 = 0$ *for all* $\lambda \leq 1$.

Proof :

$$G_{J_t}(iy) = \frac{(2-r)iy + (1/\lambda - 1) + \sqrt{C - Ay^2 - iBy}}{-2y(y+i)}$$

Thus,

$$-yG_{J_t}(iy) = \frac{(2-r)iy + \sqrt{C} + \sqrt{C - Ay^2 - iBy}}{2(y^2+1)}(y-i)$$

Since G maps $\mathbb{C}^+$ to $\mathbb{C}^-$, where $\mathbb{C}^+$ and $\mathbb{C}^-$ denote respectively the set of complex numbers with positive and negative imaginary part, then

$$-yG_{J_t}(iy) = \frac{(E(y) + iF(y))(y-i)}{2(y^2+1)}$$

in a neighbourhood of 0, where

$$\begin{aligned}
E(y) &= \sqrt{C} - \sqrt{\frac{\sqrt{(C-Ay^2)^2 + B^2y^2} + (C-Ay^2)}{2}} \\
F(y) &= (r-2)y + \sqrt{\frac{\sqrt{(C-Ay^2)^2 + B^2y^2} - (C-Ay^2)}{2}}
\end{aligned}$$

REMARK. We can easily see that $E(y) < 0$ and $F(y) > 0$ near 0. The first assertion is equivalent to $B^2 > 4CA$ which is true since $r > 1/\lambda$ (recall that $r = \lambda\theta, 0 < \theta < 1$). The second is equivalent to $B^2 > ((r-2)C)^2$ which is very easy to verify. Consequently,

$$-y\Im(G_{J_t}(iy)) = \frac{F(y)y - E(y)}{2(y^2+1)},$$

and the result follows by taking the limit. ∎

PROPOSITION 5.2. *For all* $\lambda \in [0,1]$ *and* $1/\theta \geq \lambda + 1$, *we have* $a_1 = 0$.

Proof : One has :

$$\begin{aligned}
G_{J_t}(1+iy) &= \frac{(2-r)(1+iy) + (1/\lambda - 1) + \sqrt{A(1+iy)^2 - B(1+iy) + C}}{2iy(1+iy)} \\
&= \frac{1 + 1/\lambda - r - iy(r-2) + \sqrt{A(1+iy)^2 - B(1+iy) + C}}{-2y(y-i)}
\end{aligned}$$

Now, consider the square root term, then, $(a+ib)^2 = A(1+iy)^2 - B(1+iy) + C$ is equivalent to

$$\begin{aligned}
a^2 + b^2 &= \sqrt{(A(1-y^2) + C - B)^2 + y^2(2A-B)^2} \\
a^2 - b^2 &= A(1-y^2) + C - B \\
2ab &= (2A-B)y = \left(\frac{2}{\lambda\theta}\right)^2 (2\lambda\theta^2 - \theta(\lambda+1) + 1)y
\end{aligned}$$

Note that $ab > 0 \, \forall \theta \in]0,1[$. In fact, let $f_\lambda(\theta) := 2\lambda\theta^2 - \theta(\lambda+1) + 1$, then,

$$f'_\lambda(\theta) = 4\lambda\theta - (\lambda+1) \quad \text{so that} \quad f'_\lambda(\theta) = 0 \Leftrightarrow \theta = \frac{\lambda+1}{4\lambda}$$

Since $\theta \in]0,1[$, we deduce that, if $\lambda > 1/3$, then $f_\lambda(\theta) \geq 1 - (\lambda+1)^2/8\lambda > 0$ since $8\lambda > (\lambda+1)^2$, else, $f'_\lambda(\theta) < 0$ and $f_\lambda(\theta) \geq f_\lambda(0) = 1$.
Hence, we shall take $a > 0$ and $b > 0$ since $\Im(-yG(1+iy)) > 0$. The two first

equalities give :

$$\begin{aligned} a &= \sqrt{\frac{\sqrt{(A(1-y^2)+C-B)^2+y^2(2A-B)^2}+A(1-y^2)+C-B}{2}} \\ b &= \sqrt{\frac{\sqrt{(A(1-y^2)+C-B)^2+y^2(2A-B)^2}-A(1-y^2)+C-B}{2}} \end{aligned}$$

which implies :

$$-yG_{J_t}(1+iy) = \frac{[a+(1+1/\lambda)-r+i(b-(r-2)y](y+i)}{2\pi(y^2+1)},$$

Next, we note that :

LEMMA 5.2. $A+C-B \geq 0$.

Proof :

$$\begin{aligned} A+C-B &= \left(\frac{1}{\lambda\theta}\right)^2 + \left(1-\frac{1}{\lambda}\right)^2 - \frac{2}{\lambda\theta}\left(1+\frac{1}{\lambda}\right)+\frac{4}{\lambda} \\ &= \left(\frac{1}{\lambda\theta}\right)^2 (1+\theta^2(\lambda+1)^2-2\theta(\lambda+1)) \\ &= \left(\frac{1}{\lambda\theta}\right)^2 (1-\theta(\lambda+1))^2 \qquad \blacksquare \end{aligned}$$

Taking the limit, it follows that $a_1 = \sqrt{A+C-B}-(r-1-1/\lambda)$. Finally, it remains to show that :

LEMMA 5.3. *If* $1/\theta \geq \lambda+1$, *then* $\sqrt{A+C-B} = r-(1+1/\lambda)$.

Proof : In this case, $1-\theta(\lambda+1) \geq 0$. Hence, the result follows from the previous Lemma and from :

$$r-\left(1+\frac{1}{\lambda}\right) = \frac{1}{\lambda\theta}(1-\theta(\lambda+1)) \qquad \square$$

PROPOSITION 5.3. *For all* $\lambda \in [0,1]$ *and all* $\theta \in]0,1]$, *the* J_t*'s law has a continuous part given by :*

$$g(x) = \frac{\sqrt{Bx-Ax^2-C}}{2\pi x(1-x)}$$

for some $x \in [x_-, x_+] \subset [0,1]$ *where* x_- *and* x_+ *denote the roots (when they exist) of* $Ax^2-Bx+C=0$ *.*

Proof : One has :

$$G_{J_t}(x+iy) = -\frac{[(1/\lambda-1)-(r-2)x-i(r-2)y+a+ib][x(1-x)+y^2+iy(2x-1)]}{2[(x(x-1)-y^2)^2+4y^2(1-x)^2]}$$

where $(a+ib)^2 = A(x^2-y^2) - Bx + C + i(2A-B)y$. Thus :

$$g(x) = -\lim_{y\to 0^+} \frac{1}{\pi}\Im(G_{J_t}(x+iy)) = \frac{1}{2\pi x(1-x)} \lim_{y\to 0^+} b$$

Hence, it suffices to compute the expression of $b = b(x,y)$ and find $x_-, x_+ \in]0,1[$ such that $b(x,y) > 0$ for very small y and $x \in [x_-, x_+]$. From

$$b = \sqrt{\frac{\sqrt{(A(x^2-y^2)-Bx+C)^2 + (2A-B)^2y^2} - (A(x^2-y^2)-Bx+C)}{2}},$$

we get :

$$\lim_{y\to 0^+} b(x,y) = \sqrt{\frac{\sqrt{(Ax^2-Bx+C)^2} - (Ax^2-Bx+C)}{2}},$$

Consequently,

$$\lim_{y\to 0^+} b(x,y) > 0 \Leftrightarrow Ax^2 - Bx + C < 0$$

for some $x \in [x_-, x_+]$ which is easy to see, since the infimum of $h : x \to Ax^2 - Bx + C$ is reached at $B/2A < 1$ (see Prop. 5.2) and $h(0) > 0$, $h(1) > 0$ (see Lemma 5.2), $h(B/2A) = (4AC - B^2)/4A < 0$ (Prop. 5.1).

Remark. 1/It is easy to see that $4AC - B^2 < 0$. Indeed,

$$\begin{aligned} 4AC - B^2 &= 4\left[r^2\left(1-\frac{1}{\lambda}\right)^2 - \left(r\left(1+\frac{1}{\lambda}\right) - \frac{2}{\lambda}\right)^2\right] \\ &= 16\left[\frac{r}{\lambda}\left(1+\frac{1}{\lambda}\right) - \frac{(r^2+1)}{\lambda^2}\right] \\ &= \frac{16}{\lambda^2}(r(\lambda+1) - (r^2+1)) < 0 \end{aligned}$$

for $\lambda \in [0,1]$.

5.1. Some links with Capitaine-Casalis results. In [**23**], authors proved the following result : Given two independent complex Wishart matrices X and Y with respective distributions $W(m, p(m), (1/m)I_m)$ and $W(m, q(m), (1/m)I_m)$, then the limiting distribution of the Beta matrix $Z := (X+Y)^{-1/2}X(X+Y)^{-1/2}$ is given by :

$$\nu_{\alpha,\beta}(dx) = \max(0, 1-\alpha)\delta_0(dx) + \max(0, 1-\beta)\delta_1(dx) + g(x)\mathbf{1}_{[x_-,x_+]}dx$$

where $\alpha = \lim_{m\to\infty} p(m)/m$, $\beta = \lim_{m\to\infty} q(m)/m$ and

$$\begin{aligned} \lambda_{\pm} &= \left(\sqrt{\frac{\alpha}{\alpha+\beta}\left(1-\frac{1}{\alpha+\beta}\right)} \pm \sqrt{\frac{1}{\alpha+\beta}\left(1-\frac{\alpha}{\alpha+\beta}\right)}\right)^2 \\ g(x) &= K\frac{\sqrt{(x-x-)(x_+-x)}}{x(1-x)} \end{aligned}$$

They find that :

$$G(z)=\frac{(\alpha+\beta-2)z+1-\alpha-\sqrt{(\alpha+1-(\alpha+\beta)z)^2-4\alpha(1-z)}}{2z(1-z)}$$

Comparing both results, we notice that

$$\alpha=\frac{1}{\lambda},\qquad \alpha+\beta=r,\qquad \beta=r-\frac{1}{\lambda}=\frac{1}{\lambda}\left(\frac{1}{\theta}-1\right)$$

Besides, we can see that conditions $\lambda\in[0,1], 1/\theta\geq\lambda+1$ implie that $\alpha\geq 1,\beta\geq 1$ so that $a_0=b_0=0$. On the other hand,

$$x_{\pm}=\left(\sqrt{\frac{1}{\lambda r}\left(1-\frac{1}{r}\right)}\pm\sqrt{\frac{1}{r}\left(1-\frac{1}{\lambda r}\right)}\right)^2=\left(\sqrt{\theta(1-\lambda\theta)}\pm\sqrt{\lambda\theta(1-\theta)}\right)^2$$

Hence, for $\lambda=1$, $x_-=0$ and if $\theta=1/\lambda+1$, then $x_+=1$. As a result :

Proposition 5.4. *$\forall\lambda\in]0,1], 1/\theta\geq\lambda+1\,(\theta\in]0,1/2]$ for instance) , J_t and $P-J_t$ are injective operators in the compressed space $P\mathscr{A}P$. For $\lambda\in]0,1[$ and $1/\theta>\lambda+1$, these operators are invertible in $P\mathscr{A}P$. Moreover, J_t is a solution of (48).*

Remarks. 1/In [**23**], authors omit the normalizing constant $\sqrt{A}/2\pi=(2\pi\lambda\theta)^{-1}$, however one can compute it as follows : since

$$\frac{\sqrt{(x_+-x)(x-x_-)}}{x(1-x)}=\frac{\sqrt{(x_+-x)(x-x_-)}}{x}+\frac{\sqrt{(x_+-x)(x-x_-)}}{1-x}$$

Then , the normalizing constant is given by :

$$\begin{aligned}K^{-1}&=\int_{x_-}^{x_+}\frac{\sqrt{(x_+-x)(x-x_-)}}{x}dx+\int_{1-x_+}^{1-x_-}\frac{\sqrt{(1-x_--x)(x-1+x_+)}}{x}dx\\&:=I(x_-,x_+)+I(1-x_+,1-x_-)\end{aligned}$$

Moreover, using the variable change $u=(x-x_-)/(x_+-x_-)$ and the integral representation of ${}_2F_1$ (cf [**3**]), one has :

$$\begin{aligned}I(x_-,x_+)&=\frac{(x_+-x_-)^2}{x_+}B(\frac{3}{2},\frac{3}{2})\,{}_2F_1(1,\frac{3}{2},3,\frac{x_+-x_-}{x_+})\\&\overset{(1)}{=}\frac{\pi}{4}\frac{(x_+-x_-)^2}{x_++x_-}\,{}_2F_1(\frac{1}{2},1,2,\left(\frac{x_+-x_-}{x_++x_-}\right)^2)\\&\overset{(2)}{=}\frac{\pi}{2}(x_++x_--2\sqrt{x_+x_-})\end{aligned}$$

where in (1) we used (see [**3**]) :

$${}_2F_1(a,b,2b,z)=(1-z/2)^{-a}\,{}_2F_1(a/2,(a+1)/2,b+1/2;(z/2-z)^2),\quad 0<|z|<1,$$

and in (2), we used

$$ {}_2F_1(\frac{1}{2}, 1, 2; z) = 2\frac{1-\sqrt{1-z}}{z}, \qquad 0 < |z| < 1. $$

As a result, $K^{-1} = \pi(1 - \sqrt{x_+x_-} - \sqrt{(1-x_+)(1-x_-)}) = 2\pi\lambda\theta$ (see [**106**] for another computation).
2/ In [**43**], Doumerc derived for $p(m) \geq m+1$ and $q(m) \geq m+1$ where $q(m) = d(m) - p(m)$, the following SDE for the real matrix Jacobi process :

$$ dJ_t = \sqrt{I_m - J_t} dB_t \sqrt{J_t} + \sqrt{J_t} dB_t^T \sqrt{I_m - J_t} + (p(m)I_m - (p(m) + q(m))J_t)\, dt $$

where $(B_t)_{t\geq 0}$ is a real $m \times m$ Brownian matrix. When both J_0 and $I_m - J_0$ are invertible, a strong uniqueness holds for this SDE. The complex version satisfies :

$$ dJ_t = \sqrt{I_m - J_t} dB_t \sqrt{J_t} + \sqrt{J_t} dB_t^\star \sqrt{I_m - J_t} + (p(m)I_m - (p(m) + q(m))J_t)\, dt $$

where $(B_t)_{t\geq 0}$ is a complex $m \times m$ Brownian matrix. A similar uniqueness result holds for $p(m)$, $q(m) \geq m$. Heuristically, if we consider the ratio $dJ_t/(d(m))$ and let m go to infinity, then this SDE converges weakly (up to a constant) to its free counterpart, since normalized complex Brownian matrix converges in distribution to the free complex Brownian motion. It is also worth noting that conditions $p(m) \geq m$ and $q(m) \geq m$ are in agreement with $\lambda \in [0, 1]$ and $1/\theta \geq \lambda + 1$.
3/ From a combinatorial point of view, the result of Nica and Speicher reads :

$$ \Phi(J_t^n) = \sum_{\pi \in NC(n)} k_\pi(P, \ldots, P)\theta^{n+1-|\pi|} $$

where $NC(n)$ denotes the set of non-crossing partitions of $\{1, \ldots, n\}$, $|\pi|$ is the cardinality of π and k_π is the corresponding mixed cumulant (see [**107**] for more details).

6. Free Jacobi Process : the general case

In this section, we will suppose that $\lambda \leq 1$ and $1/\theta \geq \lambda + 1$. Let $Y_0 \in \mathscr{A}$ such that $0 < J_0 := PY_0QY_0^\star P < P$, that is J_0 and $P - J_0$ are invertible in $P\mathscr{A}P$. By continuity of paths, the result of Theorem 4.2 holds for $t < T$:

$$ \begin{cases} dJ_t = & U_t \sharp dX_t + V_t \sharp dY_t + (\theta P - J_t)\, dt \\ J_0 = & PY_0QY_0^\star P, \qquad 0 < J_0 < P \end{cases} $$

where

$$\begin{aligned}
W_t &= \frac{X_t + \sqrt{-1}Y_t}{\sqrt{2}} \\
U_t &= \sqrt{\frac{\lambda\theta}{2}}(\sqrt{P-J_t}\otimes\sqrt{J_t} + \sqrt{J_t}\otimes\sqrt{P-J_t}) = \sum_{i=1}^{2} A_t^i \otimes B_t^i \\
V_t &= i\sqrt{\frac{\lambda\theta}{2}}(\sqrt{P-J_t}\otimes\sqrt{J_t} - \sqrt{J_t}\otimes\sqrt{P-J_t}) = \sum_{i=1}^{2} C_t^i \otimes D_t^i
\end{aligned}$$

and X and Y are two free $P\mathscr{A}_tP$-free-Brownian motions. Now, let us recall that for any operator $Z \in P\mathscr{A}P$, we set (see [**13**]) :

$$\begin{aligned}
\partial Z^n &= \sum_{k=0}^{n-1} Z^k \otimes Z^{n-k-1} \\
\Delta_U(Z^n) &= 2\sum_{i,j} \sum_{\substack{k,l\geq 0 \\ k+l\leq n-2}} Z^k A_t^i B_t^j Z^{n-k-l-2}\tilde{\Phi}(B_t^i Z^l A_t^j)
\end{aligned}$$

where $U = \sum_i A^i \otimes B^i$ is an adapted bi-process.

Proposition 6.1. *Let X, Y be two free free-Brownian motions, U, V be two adapted integrable bi-processes and K an adapted process in $P\mathscr{A}P$. Let*

$$dM_t = U_t\sharp dX_t + V_t\sharp dY_t + K_t dt$$

then, for every polynomial R, we have :

$$\begin{aligned}
dR(M_t) = {} & \partial R(M_t)\sharp(U_t\sharp dX_t) + \partial R(M_t)\sharp(V_t\sharp dY_t) + \partial R(M_t)\sharp K_t dt \\
& + \frac{1}{2}(\Delta_U R(M_t) + \Delta_V R(M_t))dt
\end{aligned}$$

Proof: When $V = 0\otimes 0$, this is the free Itô's formula stated in [**13**]. By linearity, it suffices to prove the formula for monomials. To do this, we shall proceed by induction. Hence, assume that :

$$\begin{aligned}
dM_t^n = {} & \partial M_t^n\sharp(U_t\sharp dX_t) + \partial M_t^n\sharp(V_t\sharp dY_t) + \partial M_t^n\sharp K_t dt + \\
& \frac{1}{2}(\Delta_U M_t^n + \Delta_V M_t^n)dt
\end{aligned}$$

By free integration by parts formula (see [**12**]), we have :

$$\begin{aligned}
dM_t^{n+1} &= d(M_t M_t^n) = dM_t\, M_t^n + M_t\, dM_t^n + (dM_t)(dM_t^n) \\
&= (1\otimes M_t^n + M_t\partial M_t^n)\sharp(U_t\sharp dX_t) + (1\otimes M_t^n + M_t\partial M_t^n)\sharp(V_t\sharp dY_t) \\
&+ (1\otimes M_t + M_t\partial M_t^n)\sharp K_t dt + \frac{1}{2}M_t(\Delta_U M_t^n + \Delta_V M_t^n)dt + (dM_t)(dM_t^n)
\end{aligned}$$

On the other hand, we can easily see that :

$$1 \otimes M_t^n + M_t \partial M_t^n = 1 \otimes M_t^n + \sum_{k=1}^{n} M_t^k \otimes M_t^{n-k} = \partial M_t^{n+1},$$

Then, using the fact that $(dX)(dY) = 0$ by the freeness of X and Y ([**79**]), we get :

$$(dM_t)(dM_t^n) = \sum_{i,j}\sum_{l=0}^{n-1} A_t^i B_t^j M_t^{n-l-1}\tilde{\Phi}(B_t^i M_t^l A_t^j) + \sum_{i,j}\sum_{l=0}^{n-1} C_t^i D_t^j M_t^{n-l-1}\tilde{\Phi}(D_t^i M_t^l C_t^j)$$

Moreover,

$$\begin{aligned} M_t \Delta_U(M_t^n) &= 2\sum_{i,j}\sum_{\substack{k,l\geq 0 \\ k+l\leq n-2}} M_t^{k+1} A_t^i B_t^j M_t^{n-k-l-2}\tilde{\Phi}(B_t^i M_t^l A_t^j) \\ &= 2\sum_{i,j}\sum_{l=0}^{n-2}\sum_{k=1}^{n-l-1} M_t^k A_t^i B_t^j M_t^{n-k-l-1}\tilde{\Phi}(B_t^i M_t^l A_t^j) \end{aligned}$$

and the same holds for

$$M_t \Delta_V(M_t^n) = 2\sum_{i,j}\sum_{l=0}^{n-2}\sum_{k=1}^{n-l-1} M_t^k C_t^i D_t^j M_t^{n-k-l-1}\tilde{\Phi}(C_t^i M_t^l D_t^j)$$

Consequently, we get :

$$\frac{1}{2} M_t(\Delta_V(M_t^n) + \Delta_U(M_t^n)) + (dM_t)(dM_t^n) = \frac{1}{2}(\Delta_V(M_t^{n+1}) + \Delta_U(M_t^{n+1})) \quad \blacksquare$$

6.1. A recurrence formula for free Jacobi moments.

Corollary 6.1. *Let $m_n(t) := \tilde{\Phi}(J_t^n)$ for $n \geq 2$ and $t < T$. Then, we have the following recurrence relation :*

$$m_n(t) = m_n(0) - n\int_0^t m_n(s)ds + n\theta\int_0^t m_{n-1}(s)ds + \lambda\theta n\sum_{k=0}^{n-2}\int_0^t m_{n-k-1}(s)(m_k(s) - m_{k+1}(s))ds$$

or equivalently,

$$\frac{dm_n(t)}{dt} = -nm_n(t) + n\theta m_{n-1}(t) + \lambda\theta n\sum_{k=0}^{n-2} m_{n-k-1}(t)(m_k(t) - m_{k+1}(t))$$

Proof : Using Proposition 6.1, we get :

$$dJ_t^n = \text{martingale} + \sum_{k=0}^{n-1} J_t^k(\theta P - J_t)J_t^{n-k-1}dt + \frac{1}{2}(\Delta_U(J_t^n) + \Delta_V(J_t^n))dt$$

Next, we compute

$$\Delta_U(J_t^n) = 2\sum_{i,j=1}^{2}\sum_{\substack{k,l\geq 0\\ k+l\leq n-2}} J_t^k A_t^i B_t^j J_t^{n-k-l-2}\tilde{\Phi}(B_t^i J_t^l A_t^j)$$

$$\overset{(1)}{=} 2\sum_{i,j=1}^{2}\sum_{\substack{k,l\geq 0\\ k+l\leq n-2}} A_t^i B_t^j J_t^{n-l-2}\tilde{\Phi}(B_t^i J_t^l A_t^j)$$

$$= 2\sum_{i,j=1}^{2}\sum_{l=0}^{n-2}(n-l-1)A_t^i B_t^j J_t^{n-l-2}\tilde{\Phi}(B_t^i J_t^l A_t^j) = 2\sum_{i,j=1}^{2}\sum_{l=0}^{n-2}(l+1)A_t^i B_t^j J_t^l\tilde{\Phi}(B_t^i J_t^{n-l-2} A_t^j)$$

$$\overset{(2)}{=} 2\sum_{l=0}^{n-2}(l+1)[\frac{\lambda\theta}{2}(P-J_t)J_t^l\tilde{\Phi}(J_t^{n-l-1}) + \frac{\lambda\theta}{2}J_t^{l+1}\tilde{\Phi}(J_t^{n-l-2}(P-J_t))$$
$$+ \lambda\theta\sqrt{P-J_t}\sqrt{J_t}J_t^l\tilde{\Phi}(J_t^{n-l-2}\sqrt{J_t}\sqrt{P-J_t})]$$

where in both (1) and (2), we used the fact that A_t^i, B_t^j and J_t commute $\forall\, i,j \in \{1,2\}$. Similarly,

$$\Delta_V(J_t^n) = 2\sum_{l=0}^{n-2}(l+1)[\frac{\lambda\theta}{2}(P-J_t)J_t^l\tilde{\Phi}(J_t^{n-l-1}) + \frac{\lambda\theta}{2}J_t^{l+1}\tilde{\Phi}(J_t^{n-l-2}(P-J_t))$$
$$- \lambda\theta\sqrt{P-J_t}\sqrt{J_t}J_t^l\tilde{\Phi}(J_t^{n-l-2}\sqrt{J_t}\sqrt{P-J_t})]$$

Thus, we have :

$$\frac{1}{2}(\Delta_U(J_t^n)+\Delta_V(J_t^n)) = \lambda\theta\sum_{l=0}^{n-2}(l+1)[(P-J_t)J_t^l\tilde{\Phi}(J_t^{n-l-1})+J_t^{l+1}\tilde{\Phi}(J_t^{n-l-2}(P-J_t))]$$

Taking the expectation, it yields

$$\frac{1}{2}\tilde{\Phi}(\Delta_U(J_t^n)+\Delta_V(J_t^n)) = \lambda\theta\times$$
$$\left(\sum_{l=0}^{n-2}(l+1)[\tilde{\Phi}((P-J_t)J_t^l)\tilde{\Phi}(J_t^{n-l-1})] + \sum_{l=0}^{n-2}(l+1)[\tilde{\Phi}(J_t^{l+1})\tilde{\Phi}(J_t^{n-l-2}(P-J_t))]\right)$$
$$= \lambda\theta\times\left(\sum_{l=0}^{n-2}(l+1)[\tilde{\Phi}((P-J_t)J_t^l)\tilde{\Phi}(J_t^{n-l-1})] + \sum_{l=0}^{n-2}(n-l-1)[\tilde{\Phi}(J_t^{n-l-1})\tilde{\Phi}(J_t^l(P-J_t))]\right)$$
$$= n\lambda\theta\sum_{l=0}^{n-2}[\tilde{\Phi}((P-J_t)J_t^l)\tilde{\Phi}(J_t^{n-l-1})]$$
$$= n\lambda\theta\sum_{l=0}^{n-2}[m_{n-l-1}(t)(m_l(t)-m_{l+1}(t)]$$

Furthermore, since $P - J_t$ and J_t commute, we can easily see that :

$$\tilde{\Phi}\left(\sum_{k=0}^{n-1} J_t^k(\theta P - J_t)J_t^{n-k-1}\right) = n\tilde{\Phi}(J_t^{n-1}(\theta P - J_t)) \qquad \Box$$

PROPOSITION 6.2. *If J_0 and $P - J_0$ are invertible in $P\mathscr{A}P$, then for all $\lambda \in]0,1]$, $1/\theta \geq 1+\lambda$ and $t \geq 0$, $P - J_t$ and J_t are injective operators in $P\mathscr{A}P$.*

Proof : It is known that for a self - adjoint operator $a \in P\mathscr{A}P$ such that $0 < a < P$,

$$\log(P-a) = -\sum_{n=1}^{\infty} \frac{a^n}{n}$$

Since $\tilde{\Phi}(a^n) = \int x^n \mu(dx)$ for a positive compactly supported measure μ, we get :

$$\tilde{\Phi}(\log(P-a)) = -\sum_{n=1}^{\infty} \frac{\tilde{\Phi}(a^n)}{n} = -\tilde{\Phi}(a) - \sum_{n=2}^{\infty} \frac{\tilde{\Phi}(a^n)}{n}$$

Thus, substituting the moments of J_t, one has (Corollary 6.1) for all $t < T$:

$$\tilde{\Phi}(\log(P-J_t)) = \tilde{\Phi}(\log(P-J_0)) - \theta t + \int_0^t \tilde{\Phi}\left(\sum_{n=1}^{\infty} J_s^n\right) ds - \theta \int_0^t \tilde{\Phi}\left(\sum_{n=1}^{\infty} J_s^n\right) ds$$

$$- \lambda\theta \int_0^t \sum_{n=2}^{\infty}\sum_{k=0}^{n-2} \tilde{\Phi}(J_s^{n-1-k})\tilde{\Phi}(J_s^k(P-J_s))ds$$

$$= \tilde{\Phi}(\log(P-J_0)) - \theta t + (1-\theta)\int_0^t \tilde{\Phi}(J_s(P-J_s)^{-1})ds - \lambda\theta \int_0^t \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} \tilde{\Phi}(J_s^n)\tilde{\Phi}(J_s^k(P-J_s))ds$$

$$= \tilde{\Phi}(\log(P-J_0)) - \theta t + (1-\theta)\int_0^t \tilde{\Phi}(J_s(P-J_s)^{-1})ds - \lambda\theta \int_0^t \tilde{\Phi}(J_s(P-J_s)^{-1})\tilde{\Phi}(P)ds$$

$$= \tilde{\Phi}(\log(P-J_0)) - \theta t + (1-\theta-\lambda\theta)\int_0^t \tilde{\Phi}(J_s(P-J_s)^{-1})ds$$

$$= \tilde{\Phi}(\log(P-J_0)) - (1-\lambda\theta)t + (1-\theta-\lambda\theta)\int_0^t \tilde{\Phi}((P-J_s)^{-1})ds$$

When $\lambda \in]0,1]$ and $1/\theta \geq 1+\lambda$ then $1-\theta-\lambda\theta \geq 0$ and $1-\lambda\theta \geq 0$. Hence if $P-J_0$ is invertible, then :

$$\tilde{\Phi}(\log(P-J_t)) + (1-\lambda\theta)t \geq \tilde{\Phi}(\log(P-J_0)) > -\infty \qquad \forall\, t < T$$

which gives the injectivity of $P-J_t$, $\forall t \geq 0$. The second assertion follows since $P-J$ is a $FJP(\lambda\theta/(1-\theta), 1-\theta)$ and since J_0 is invertible. Indeed, $1/\theta \geq \lambda+1 \Rightarrow \lambda\theta/(1-\theta) \leq 1$ and $\lambda \leq 1 \Rightarrow (\lambda-1)\theta \leq 0 \Rightarrow (\lambda\theta)/(1-\theta)+1 = (\theta(\lambda-1)+1)/(1-\theta) \leq 1/(1-\theta)$. Thus, similar computations applies when replacing J_t by $P-J_t$. ■

Remark. 1/ One can also recover Rouault's result on $\widetilde{\Phi}(\log(J_t))$ (see [**106**]). Let $0 \leq z \leq 1$ and $\lambda \in]0,1[, 1/\theta > \lambda + 1$. Then, one can see that :

$$-\frac{d}{dz}\tilde{\Phi}(\log(P - zJ_t)) = -\frac{1}{z}\left(\frac{1}{z}G_{J_t}(\frac{1}{z}) - 1\right) = \frac{(1+1/\lambda)z - r + \sqrt{Cz^2 - Bz + A}}{2z(1-z)}$$

Note that this derivative is well defined for $z = 0$ and $z = 1$. It follows that :

$$(49) \qquad 2\tilde{\Phi}(\log(P - J_t)) = -\int_0^1 \frac{(1+1/\lambda)z - r + \sqrt{Cz^2 - Bz + A}}{z(1-z)}dz$$

Note first that $Cz^2 - Bz + A > 0 \,\forall \lambda \in]0,1[, 1/\theta > \lambda + 1$ since $x_+, x_- \in]0,1[$ are the roots of $Az^2 - Bz + C$ (so that $z < 1/x_+$) . In order to evaluate the integral in the right, we use the variable change $\sqrt{A}(1 - uz) = \sqrt{Cz^2 - Bz + A}$, which gives :

$$z = \frac{2Au - B}{Au^2 - C},\ 1 - z = \frac{Au^2 - 2Au + B - C}{Au^2 - C},\ dz = -2A\frac{Au^2 - Bu + C}{(Au^2 - C)^2}du.$$

Moreover, since $A - B + C = A(1 - \theta(\lambda+1))^2 \geq 0$ and $\theta(1+\lambda) \leq 1$, then the roots of $Au^2 - 2Au + B - C = 0$ are given by : $u_\pm = 1 \pm (1 - \theta(\lambda + 1))$. On the other hand, $B/2A = (1/2)(x_+ + x_-) = \theta(\lambda + 1 - 2\lambda\theta)$. Hence our expression factorizes into :

$$\begin{aligned}
\tilde{\Phi}(\log(P - J_t)) &= -\frac{1}{\sqrt{A}}\int_{B/2A}^{\theta(\lambda+1)} \frac{(u - \theta(\lambda+1))(Au^2 - Bu + C)}{(u^2 - C/A)(Au^2 - 2Au + B - C)}du \\
&= -\frac{1}{\sqrt{A}}\int_{B/2A}^{\theta(\lambda+1)} \frac{(Au^2 - Bu + C)}{(u^2 - C/A)(u - u_+)}du \\
&= \int_{B/2A}^{\theta(\lambda+1)} \frac{C_1}{u - \theta(1-\lambda)} + \frac{C_2}{u + \theta(1-\lambda)} + \frac{C_3}{u - u_+}du
\end{aligned}$$

for some constants C_1, C_2, C_3 depending on both λ, θ, given by :

$$C_1 \;=\; 1, \qquad C_2 = 1/\lambda, \qquad C_3 = \frac{1 - \theta(\lambda+1)}{\lambda\theta}$$

Thus, one gets :

$$\begin{aligned}
\tilde{\Phi}(\log(P - J_t)) &= -\left[C_1\log(u - \theta(1-\lambda)) + C_2\log(u + \theta(1-\lambda)) + C_3\log(u_+ - u)\right]_{B/2A}^{\theta(\lambda+1)} \\
&= \log(1-\theta) + \frac{1}{\lambda}\log(1 - \lambda\theta) - C_3\log\left[\frac{(1 - \theta(\lambda+1))}{1 - \theta(\lambda+1) + \lambda\theta^2}\right] \\
&= (1 + C_3)\log(1-\theta) + (\frac{1}{\lambda} + C_3)\log(1 - \lambda\theta) - C_3\log(1 - \theta(\lambda+1)) \\
&= \frac{(1-\theta)\log(1-\theta) + (1-\lambda\theta)\log(1-\lambda\theta) - (1 - \theta(\lambda+1))\log(1 - \theta(\lambda+1))}{\lambda\theta}
\end{aligned}$$

Note that the result extends for all $\lambda \in]0,1]$, $1/\theta \geq \lambda + 1$. Since $P - J$ is still a $FJP(\lambda\theta/(1-\theta), 1-\theta)$, then :

$$\tilde{\Phi}(\log(J_t)) = \frac{\theta \log \theta + (1-\lambda\theta)\log(1-\lambda\theta) - \theta(1-\lambda)\log(\theta(1-\lambda))}{\lambda\theta} \qquad \blacksquare$$

COROLLARY 6.2. *Under the same conditions of Proposition 6.2, the $FJP(\lambda, \theta)$ satisfies for all $t \geq 0$ the following SDE :*

$$dJ_t = \sqrt{\lambda\theta}\sqrt{P - J_t}dW_t\sqrt{J_t} + \sqrt{\lambda\theta}\sqrt{J_t}dW_t^\star\sqrt{P - J_t} + (\theta P - J_t)\,dt$$

where W is a complex free Brownian motion.

6.2. Free martingales polynomials. In this paragraph, we consider a stationary $FJP(1, 1/2)$ starting at J_0, the law of which is the Beta law $B(1/2, 1/2)$. Recall that a $\mathscr{A}_t$-adapted free process $(X_t)_{t\geq 0}$ is a $\mathscr{A}_t$-free martingale if and only if $\Phi(X_t|\mathscr{A}_s) = X_s$ (see [**2**],[**12**], [**15**]).

PROPOSITION 6.3. *Let $\mathscr{J}_t$ denotes the von Neumann subalgebra generated by $(J_s,\, s \leq t)$ and let $0 < r < 1$. Then, the process $R_t := ((1+re^t)P - 2re^tJ_t)((1+re^t)^2P - 4re^tJ_t)^{-1})_{t<-\ln r}$ is a $\mathscr{J}_t$-free martingale.*

Proof : R_t can be written as :

$$\begin{aligned} R_t &= \left[\frac{P}{1+re^t} - 2\frac{re^t}{(1+re^t)^2}J_t\right]\left[P - \frac{4re^t}{(1+re^t)^2}J_t\right]^{-1} \\ &= \left[\frac{1-re^t}{2(1+re^t)}P + \frac{1}{2}\left(P - \frac{4re^t}{(1+re^t)^2}J_t\right)\right]\left[P - \frac{4re^t}{(1+e^t)^2}J_t\right]^{-1} \\ &:= \frac{1-re^t}{2(1+re^t)}H_t + \frac{P}{2} \end{aligned}$$

where

$$H_t = \left[P - \frac{4re^t}{(1+re^t)^2}J_t\right]^{-1} = \sum_{n\geq 0}\frac{(4re^t)^n}{(1+re^t)^{2n}}J_t^n$$

since $4re^t < (1+re^t)^2$ and $0 \leq J_t \leq P$ for all $t > 0$. It follows that :

$$2dR_t = \frac{1-re^t}{1+re^t}dH_t - \frac{2re^t}{(1+re^t)^2}H_t dt$$

On the other hand, one has for $1 \leq l \leq n-1$ and $n \geq 2$:

$$\tilde{\Phi}(J_t^{n-l}) = \frac{\Gamma(n-l+1/2)}{\sqrt{\pi}(n-l)!}, \qquad \tilde{\Phi}(J_t^{n-l-1}(P - J_t)) = \frac{\Gamma(n-l-1/2)}{2\sqrt{\pi}(n-l)!}$$

From the proof of Proposition 6.1, we deduce that, for all $t > 0$:

$$
\begin{aligned}
dJ_t^n &= M_t + n(\frac{P}{2} - J_t)J_t^{n-1}dt + \frac{1}{2}(\Delta_U(J_t^n) + \Delta_V(J_t^n)) \\
&= M_t + \left[n(\frac{P}{2} - J_t)J_t^{n-1} + \frac{1}{2}\sum_{l=1}^{n-1} l[(P - J_t)J_t^{l-1}\tilde{\Phi}(J_t^{n-l}) + J_t^l\tilde{\Phi}(J_t^{n-l-1}(P - J_t))]\right] dt \\
&= M_t + \left[n(\frac{P}{2} - J_t)J_t^{n-1} + \sum_{l=1}^{n-1} l \left[(P - J_t)J_t^{l-1}\frac{\Gamma(n-l+1/2)}{2\sqrt{\pi}(n-l)!} + \frac{J_t^l}{2}\frac{\Gamma(n-l-1/2)}{2\sqrt{\pi}(n-l)!}\right]\right] dt \\
&= M_t + \left[n(\frac{P}{2} - J_t)J_t^{n-1} + \sum_{l=1}^{n-1} \left[l\frac{\Gamma(n-l+1/2)}{2\sqrt{\pi}(n-l)!}J_t^{l-1} - l(n-l-1)\frac{\Gamma(n-l-1/2)}{2\sqrt{\pi}(n-l)!}J_t^l\right]\right] dt \\
&= M_t + \left[n(\frac{P}{2} - J_t)J_t^{n-1} + \left[\sum_{l=1}^{n-1} l\frac{\Gamma(n-l+1/2)}{2\sqrt{\pi}(n-l)!}J_t^{l-1} - \sum_{l=2}^{n-1}\frac{(l-1)(n-l)\Gamma(n-l+1/2)}{2\sqrt{\pi}(n-l+1)!}J_t^{l-1}\right]\right] c \\
&= M_t + \left[n(\frac{P}{2} - J_t)J_t^{n-1} + \frac{1}{2\sqrt{\pi}}\left[\sum_{l=2}^{n-1} n\frac{\Gamma(n-l+1/2)}{(n-l+1)!}J_t^{l-1} + \frac{\Gamma(n-1/2)}{(n-1)!}P\right]\right] dt \\
&= M_t + \left[n(\frac{P}{2} - J_t)J_t^{n-1} + \frac{n}{2\sqrt{\pi}}\sum_{l=1}^{n-1}\frac{\Gamma(n-l+1/2)}{(n-l+1)!}J_t^{l-1}\right] dt \\
&= M_t - nJ_t^n dt + \frac{n}{2\sqrt{\pi}}\sum_{l=1}^{n}\frac{\Gamma(n-l+1/2)}{(n-l+1)!}J_t^{l-1}\, dt
\end{aligned}
$$

where M_t stands for the martingale part. Note that this is true for $n = 1$. Thus :

$$
\begin{aligned}
FV(dH_t) &= \sum_{n\geq 0}\frac{(4re^t)^n}{(1+re^t)^{2n}}FV(dJ_t^n) + \frac{1-re^t}{1+re^t}\sum_{n\geq 0}\frac{n(4re^t)^n}{(1+re^t)^{2n}}J_t^n dt \\
&= -\frac{2re^t}{1+re^t}\sum_{n\geq 0}\frac{n(4re^t)^n}{(1+re^t)^{2n}}J_t^n dt + \frac{1}{2\sqrt{\pi}}\sum_{n\geq 0}\frac{n(4re^t)^n}{(1+re^t)^{2n}}\sum_{l=1}^{n}\frac{\Gamma(n-l+1/2)}{(n-l+1)!}J_t^{l-1}\, dt \\
&= -\frac{2re^t}{1+re^t}\sum_{n\geq 0}\frac{n(4re^t)^n}{(1+re^t)^{2n}}J_t^n dt + \frac{1}{2}\sum_{l\geq 1}\frac{(4re^t)^l}{(1+re^t)^{2l}}\sum_{n\geq 0}\frac{(n+l)(1/2)_n}{(n+1)!}\frac{(4re^t)^n}{(1+re^t)^{2n}}J_t^{l-1}\, dt \\
&= -\frac{2re^t}{1+re^t}\sum_{n\geq 0}\frac{n(4re^t)^n}{(1+re^t)^{2n}}J_t^n dt + \frac{2re^t}{(1+re^t)^2}\sum_{l\geq 0}\frac{(4re^t)^l}{(1+re^t)^{2l}}\sum_{n\geq 0}\frac{(n+l+1)(1/2)_n}{(n+1)!}\frac{(4re^t)^n}{(1+re^t)^{2n}}J_t^l\, d \\
&= -\frac{2re^t}{1+re^t}\sum_{n\geq 0}\frac{n(4re^t)^n}{(1+re^t)^{2n}}J_t^n dt + \frac{2re^t}{(1+re^t)^2}\sum_{l\geq 0}\frac{(4re^t)^l}{(1+re^t)^{2l}}J_t^l\sum_{n\geq 0}\frac{(1/2)_n}{n!}\frac{(4re^t)^n}{(1+re^t)^{2n}}\, dt
\end{aligned}
$$

$$+\frac{1}{2}\sum_{l\geq 0}\frac{l(4re^t)^l}{(1+re^t)^{2l}}J_t^l\sum_{n\geq 0}\frac{(1/2)_n}{(n+1)!}\frac{(4re^t)^{n+1}}{(1+re^t)^{2n+2}}\,dt$$

where FV stands for the finite variation part. From (see [**3**]) :

$${}_1\mathscr{F}_0(a,z) := \sum_{p=0}^{\infty}(a)_p\frac{z^p}{p!} = (1-z)^{-a},\ |z|<1,$$

one has for all t such that $re^t < 1$:

$$\sum_{n\geq 0}\frac{(1/2)_n}{n!}\frac{(4re^t)^n}{(1+re^t)^{2n}}\,dt = \left(1-\frac{4re^t}{(1+re^t)^2}\right)^{-1/2} = \frac{1+re^t}{1-re^t}$$

and similarly, from $\int_0^u dz/\sqrt{1-z} = 2-2\sqrt{1-u}$,

$$\frac{1}{2}\sum_{n\geq 0}\frac{(1/2)_n}{(n+1)!}\frac{(4re^t)^{n+1}}{(1+re^t)^{2n+2}} = 1-\left(1-\frac{4re^t}{(1+re^t)^2}\right)^{1/2} = \frac{2re^t}{1+re^t}$$

The result follows from an easy computation. ■

COROLLARY 6.3. *Let T_k denotes the k^{th}-Tchebycheff polynomial of the first kind :*

$$T_k(x) := \cos(k\arccos(x)),\ k\geq 0,\ x\in]-1,1[$$

Thus the process $S(k)$ defined by $S_t(k) := e^{kt}T_k(2J_t - P)$ is a $\mathscr{J}_t$-free martingale.

Proof : Let us first point out the reader that these polynomials are orthogonal with respect to Beta distribution $B(1/2,1/2)$ which is the law of $FJP(1,1/2)$. The proof is standard (see [**12**] for the additive free BM) and uses the generating function of $(T_k)_{k\geq 0}$ which is given by ([**3**]) :

$$L(x,z) := \sum_{k=0}^{\infty}T_k(x)z^k = \frac{1-zx}{1-2zx+z^2}, \quad ,|z|<1$$

Letting $z = re^t$ with $0<r<e^{-t}<e^{-s}$ for $s<t$, then both $L(2J_t - P, re^t)$ and $L(2J_s - P, re^s)$ converge and an easy computation shows that :

$$L(2J_t - P, re^t) = R_t \quad \text{is a } \mathscr{J}_t - \text{free martingale.}$$

thus,

$$\tilde{\Phi}(R_t|\mathscr{J}_s) = R_s \Leftrightarrow \tilde{\Phi}[L(2J_t - P, re^t)|\mathscr{J}_s] = L(2J_s - P, re^s)$$

$\Leftrightarrow$

$$\sum_{k=0}^{\infty}\tilde{\Phi}(e^{kt}T_k(2J_t - P)|\mathscr{J}_s)r^k = \sum_{k=0}^{\infty}T_k(2J_s - P)e^{ks}r^k$$

Taking the derivative of both sides at $r=0$, we are done.

7. On The Cauchy transform of the free Jacobi process

The general hypothesis we made to derive (48) is injectivity of both J and $P - J$. It is closely related to the weights of the Dirac masses involved in the spectral measure. These weights are recovered from the Cauchy transform as stated in section 5 :

$$a_0 = \lim_{y \to 0^+} -y\Im[G(iy)], \quad a_1 = \lim_{y \to 0^+} -y\Im[G(1+iy)]$$

Hopelessly, as the curious reader can guess, we are not able to do this due to the unboundness of some terms obstructing the intertwining of limit and integral signs. First, one has from (48) and under assumptions of the previous section :

$$\tilde{\Phi}(J_t) = (\tilde{\Phi}(J_0) - \theta)e^{-t} + \theta$$

Then, a similar computation as in Proposition 6.2 using Corollary 6.1 gives for all u in the unit disk :

$$\tilde{\Phi}(\log(P - uJ_t)) = -\sum_{n \geq 1} \frac{u^n}{n}\tilde{\Phi}(J_t^n) = -u\tilde{\Phi}(J_t) - \sum_{n \geq 2} \frac{u^n}{n}\tilde{\Phi}(J_t^n)$$

$$= u(\tilde{\Phi}(J_0) - \tilde{\Phi}(J_t)) + \tilde{\Phi}(\log(P - uJ_0)) + \int_0^t \tilde{\Phi}\left(\sum_{n \geq 2}(uJ_s)^n\right) ds - \theta u \int_0^t \tilde{\Phi}\left(\sum_{n \geq 2}(uJ_s)^{n-1}\right) ds$$

$$- \lambda\theta \int_0^t \sum_{n \geq 2}\sum_{k=0}^{n-2} u^n \tilde{\Phi}(J_s^{n-k-1})\tilde{\Phi}(J_s^k(P - J_s))ds$$

$$= u(\tilde{\Phi}(J_0) - \theta)(1 - e^{-t}) + \tilde{\Phi}(\log(P - uJ_0)) + \int_0^t \tilde{\Phi}(u^2 J_s^2 (P - uJ_s)^{-1})ds$$

$$- \theta u \int_0^t \tilde{\Phi}(uJ_s(P - uJ_s)^{-1})ds - \lambda\theta u \int_0^t \tilde{\Phi}\left(\sum_{n \geq 0}(uJ_s)^{n+1}\right)\tilde{\Phi}\left(\sum_{k \geq 0}(uJ_s)^k(P - J_s)\right) ds$$

Using the fact that $u^2 J_s^2 = (uJ_s - P)(uJ_s + P) + P$ and $uJ_s = (uJ_s - P) + P$, we get :

$$\tilde{\Phi}(\log(P - uJ_t)) = u(\tilde{\Phi}(J_0) - \theta)(1 - e^{-t}) + \tilde{\Phi}(\log(P - uJ_0)) + (1 - \theta u) \int_0^t \tilde{\Phi}((P - uJ_s)^{-1})ds$$

$$+ \theta ut - \int_0^t \tilde{\Phi}((P + uJ_s))ds - \lambda\theta u \int_0^t \tilde{\Phi}(uJ_s(P - uJ_s)^{-1})\tilde{\Phi}((P - J_s)(P - uJ_s)^{-1})ds$$

$$= \tilde{\Phi}(\log(P - uJ_0)) + (1 - \theta u) \int_0^t \tilde{\Phi}((P - uJ_s)^{-1})ds - t$$

$$- \lambda\theta u \int_0^t [\tilde{\Phi}((P - uJ_s)^{-1}) - 1][1 + (u - 1)\tilde{\Phi}(J_s(P - uJ_s)^{-1})]ds$$

$$= \tilde{\Phi}(\log(P - uJ_0)) + (1 - \theta u - \lambda\theta u) \int_0^t \tilde{\Phi}((P - uJ_s)^{-1}) ds - (1 - \lambda\theta u)t$$

$$- \lambda\theta(u-1) \int_0^t \tilde{\Phi}((P - uJ_s)^{-1})\tilde{\Phi}(uJ_s(P - uJ_s)^{-1}) ds + \lambda\theta(u-1) \int_0^t \tilde{\Phi}(uJ_s(P - uJ_s)^{-1}) ds$$

$$= \tilde{\Phi}(\log(P - uJ_0)) + (1 - \theta u - \lambda\theta u + 2\lambda\theta(u-1)) \int_0^t \tilde{\Phi}((P - uJ_s)^{-1}) ds - (1 - \lambda\theta u)t$$

$$- \lambda\theta(u-1)t - \lambda\theta(u-1) \int_0^t \tilde{\Phi}^2((P - uJ_s)^{-1}) ds$$

$$= \tilde{\Phi}(\log(P - uJ_0)) + (1 - \theta u + \lambda\theta(u-2)) \int_0^t \tilde{\Phi}((P - uJ_s)^{-1}) ds - (1 - \lambda\theta)t$$

$$- \lambda\theta(u-1) \int_0^t \tilde{\Phi}^2((P - uJ_s)^{-1}) ds$$

Setting $h_t(u) = \tilde{\Phi}((P - uJ_t)^{-1})$, then $h_t(u) = (1/u)G_{J_t}(1/u) := (1/u)G_t(1/u)$ and

$$-\frac{d}{du}\tilde{\Phi}(\log(P - uJ_t)) = \tilde{\Phi}(J_t(P - uJ_t)^{-1}) = \frac{1}{u}(h_t(u) - 1)$$

Thus :

$$h_t(u) = h_0(u) + \theta(1-\lambda) \int_0^t u h_s(u) ds - (1 - \theta u + \lambda\theta(u-2)) \int_0^t u h_s'(u) ds$$
$$+ \lambda\theta \int_0^t u h_s^2(u) ds + 2\lambda\theta(u-1) \int_0^t u h_s(u) h_s'(u) ds$$

or equivalently :

$$G_t(1/u) = G_0(1/u) + \theta(1-\lambda)u \int_0^t G_s(1/u) ds + \lambda\theta \int_0^t G_s^2(1/u) ds + (1 - \theta u + \lambda\theta(u-2)) \times$$
$$\int_0^t \left[G_s(1/u) + \frac{1}{u}G_s'(1/u)\right] ds + 2\lambda\theta \left(\frac{1-u}{u^2}\right) \int_0^t \left[uG_s^2(1/u) + G_s'(1/u)G_s(1/u)\right]$$

$$= G_0(1/u) + (1 - 2\lambda\theta) \int_0^t G_s(1/u) ds + \lambda\theta \left(\frac{2}{u} - 1\right) \int_0^t G_s^2(1/u) ds$$
$$+ \frac{1 - \theta u + \lambda\theta(u-2)}{u} \int_0^t G_s'(1/u) ds + \frac{2\lambda\theta(1-u)}{u^2} \int_0^t G_s(1/u) G_s'(1/u) ds$$

As a consequence, G satisfies the p. d. e. :

Proposition 7.1.

$$G_t(z) = G_0(z) + (1 - 2\lambda\theta) \int_0^t G_s(z) ds + \lambda\theta(2z - 1) \int_0^t G_s^2(z) ds$$
$$+ ((1 - 2\lambda\theta)z - \theta(1-\lambda)) \int_0^t G_s'(z) ds + 2\lambda\theta z(z-1) \int_0^t G_s(z) G_s'(z) ds$$

Remark. The expression above can be derived in a similar way by multiplying both sides of the recurrence formula in Corollary (6.1) by u^n and summing over n. Besides, it takes the p.d.e form :

$$\partial_t G_t(z) = \partial_z\{[(1-2\lambda\theta)z - \theta(1-\lambda)]G_t(z) + \lambda\theta z(z-1)G_t^2(z)\}$$

In the stationary case, one can see that $[(1-2\lambda\theta)z - \theta(1-\lambda)]G(z) + \lambda\theta z(z-1)G^2(z) = -\lambda\theta = -1/r$ with $G := G_{J_t}$ derived in section 5. Thus the p.d.e. is satisfied.

8. Conclusion and open questions

The curious reader can check after browsing chapter 3 in [**43**] that the study of the free Jacobi process is more handable than the one of its matrix analog, in the sense that, though both cases belong to a non commutative context, more precise results on the law are derived in the infinite dimensional case, namely, the recurrence formula for the moments and the Cauchy transform though the nonlinear p.d.e. it satisfies. Nevertheless, we can not prove uniqueness of the solution of (48) as done for the matrix Jacobi process and for the free Wishart process as well. When the SDE (or free) is driven by Hölder-continuous coefficient operators, this uses mainly an invertibility argument as well as Gronwall Lemma. Hence, this can be done in the stationary case for $\lambda \in]0,1[$, $1/\theta > \lambda + 1$ (see Proposition 5.4). A general result is still an open problem.

CHAPITRE 7

Large deviations for statistics of Jacobi process

This paper is submitted in SPA.

This paper is aimed to derive large deviations for statistics of Jacobi process already conjectured by M. Zani in her Thesis. To proceed, we write in a more simple way the Jacobi semi-group density. Being given by a bilinear sum involving Jacobi polynomials, it differs from Hermite and Laguerre cases by the quadratic form of its eigenvalues. Our attempt relies on subordinating the process using a suitable random time-change. This will give an analogue of Mehler formula whence we can recover the desired expression by inverting some Laplace transforms. Once we did, an adaptation of Zani's result in the non-steepness case will provide the required large deviations principle.

1. Introduction

The Jacobi process is a Markov process on $[-1,1]$ given by the following infinitesimal generator :

$$\mathscr{L} = (1-x^2)\frac{\partial^2}{\partial^2 x} + (px+q)\frac{\partial}{\partial x}, \qquad x \in [-1,1]$$

for some real p, q, defined up to the first time when it hits the boundary. It appears as an interest rate model in finance (see [**35**]) and in genetics ([**49**]). One of the important feature is that it belongs to the class of diffusions associated to some families of orthogonal polynomials, i.e. the infinitesmal generator admits an orthogonal polynomials basis as eigenfunctions ([**6**]) such as Hermite, Laguerre and Jacobi polynomials . More precisely, if $P_n^{\alpha,\beta}$ denotes the Jacobi polynomial with parameters $\alpha, \beta > -1$ defined by :

$$P_n^{\alpha,\beta}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(-n, n+\alpha+\beta+1, \alpha+1; \frac{1-x}{2}\right), \qquad x \in [-1,1],$$

then we can see that :

$$\mathscr{L}P_n^{\alpha,\beta} = -n(n+\alpha+\beta+1)P_n^{\alpha,\beta}$$

for $p = -(\beta+\alpha+2)$ and $q = \beta - \alpha$. The semi group density of the process first appeared in [**73**] then in [**114**] where the author solved the forward Kolmogorov or Fokker-Planck equation (see [**114**], [**101**])

$$\partial_y^2[B(y)p] - \partial_y[A(y)p] = \partial_t p, \quad p = p_t(x,y),$$

where B, A are polynomials of degree 2, 1 respectively, and gave the principal solution ($p_0(x,y) = \delta_x(y)$) using the classical Sturm-Liouville theory. This gives rise to a class of stationary Markov processes satisfying :

$$\lim_{t\to\infty} p_t(x,y) = \int_{x_1}^{x_2} W(x)p_t(x,y)dx = W(y) \tag{50}$$

where W is the density function solution of the corresponding Pearson equation ([**114**]). In our case, p_t has the discrete spectral decomposition :

$$p_t(x,y) = \left(\sum_{n\geq 0}(R_n)^{-1}e^{-\lambda_n t}P_n^{\alpha,\beta}(x)P_n^{\alpha,\beta}(y)\right)W(y), \qquad x,y\in[-1,1] \tag{51}$$

where

$$\lambda_n = n(n+\alpha+\beta+1), \qquad W(y) = \frac{(1-y)^\alpha(1+y)^\beta}{2^{\alpha+\beta+1}B(\alpha+1,\beta+1)}$$

with B denoting the Beta function and[1] ([**3**], p. 99) :

$$R_n = ||P_n^{\alpha,\beta}||^2_{L^2([-1,1],W(y)dy)} = \frac{\Gamma(\alpha+\beta+2)}{2n+\alpha+\beta+1}\frac{(\alpha+1)_n(\beta+1)_n}{\Gamma(\alpha+\beta+n+1)n!}$$

Interested in total positivity, Karlin and McGregor showed the positivity of this kernel for $\alpha,\beta > -1$ ([**73**]). Few years later, Gasper ([**57**]) showed that, for $\alpha,\beta \geq -1/2$, this bilinear sum is the transition kernel of a diffusion and that is a solution of the heat equation governed by a Jacobi operator, generalizing a previous result of Bochner for ultraspherical polynomials ([**17**]). However, Gasper's intention was to study measure convolutions with respect to the kernel. It is worthnoting that λ_n has a quadratic form while in the Hermite (Brownian) and Laguerre (squared Bessel) cases $\lambda_n = n$. Hence, we will try to subordinate the Jacobi process by the mean of a random time-change in order to get a Mehler type formula. What is quite interesting is that subordinated Jacobi process semi-group, say $q_t(x,y)$, is the Laplace transform of $p_{2/t}(x,y)$. Thus, we deduce an expression for $p_t(x,y)$ by inverting some Laplace transforms already computed by Biane, Pitman and Yor (see [**16**], [**96**]). This expression, more handable than (51) will allow us to compute the normalized cumulant generating function, and then to derive a LDP for the maximum likelihood estimate (MLE) for p in the ultraspherical case, i. e. $q = 0\,(\beta=\alpha)$, a fact conjectured by Zani in her thesis. Then, using a skew product representation of the Jacobi process involving squared Bessel processes, we construct a family $\{\hat{\nu}_t\}_t$ of estimators for the index ν of the squared Bessel

[1]$(P_n^{\alpha,\beta}(x))_{n\geq 0}$ are normalized such that they form an orthogonal basis with respect to the probability measure $W(y)dy$ which is not the same used in [**3**].

process based on a Jacobi trajectory observed till time t. This satisfies a LDP with the same rate function derived for the MLE based on a squared Bessel trajectory.

1.1. Inverse Gaussian subordinator. By an *inverse Gaussian subordinator*, we mean the process of the first hitting time of a Brownian motion with drift $B_t^\mu := B_t + \mu t$, $\mu \in \mathbb{R}$, namely,

$$T_t^{\mu,\delta} = \inf\{s > 0; \quad B_s^\mu = \delta t\}, \qquad \delta > 0.$$

Using martingale methods, we can show that for each $t > 0$, $u \geq 0$,

$$\mathbb{E}(e^{-uT_t^{\mu,\delta}}) = e^{-t\delta(\sqrt{2u+\mu^2}-\mu)}$$

whence the density f_t of $T_t^{\mu,\delta}$ writes ([**1**]) :

$$f_t(s) = \frac{\delta t}{\sqrt{2\pi}} e^{\delta t\mu} s^{-3/2} \exp\left(-\frac{1}{2}(\frac{t^2\delta^2}{s} + \mu^2 s)\right) \mathbf{1}_{\{s>0\}} \tag{52}$$

1.2. The subordinated Jacobi Process. Let us consider a Jacobi process $(X_t)_{t\geq 0}$. Then the semi-group of the subordinated Jacobi process $(X_{T_t^{\mu,\delta}})_{t\geq 0}$ is given by :

$$\begin{aligned} q_t(x,y) &= \int_0^\infty p_s(x,y) f_t(s) ds \\ &= W(y) \sum_{n\geq 0} (R_n)^{-1} \left(\int_0^\infty e^{-\lambda_n s} f_t(s) ds\right) P_n^{\alpha,\beta}(x) P_n^{\alpha,\beta}(y) \\ &= W(y) \sum_{n\geq 0} (R_n)^{-1} \mathbb{E}(e^{-\lambda_n T_t^{\mu,\delta}}) P_n^{\alpha,\beta}(x) P_n^{\alpha,\beta}(y) \end{aligned}$$

Writing $\lambda_n = (n+\gamma)^2 - \gamma^2$ where $\gamma = \dfrac{\alpha+\beta+1}{2}$, and substituting $\delta = 1/\sqrt{2}$, $\mu = \sqrt{2}\gamma$ for $\alpha + \beta > -1$ in the expression of f_t, one gets :

$$\mathbb{E}(e^{-\lambda_n T_t^{\mu,\delta}}) = e^{-nt}$$

so that

$$q_t(x,y) = W(y) \sum_{n\geq 0} (R_n)^{-1} e^{-nt} P_n^{\alpha,\beta}(x) P_n^{\alpha,\beta}(y)$$

The last sum has been already computed ([**3**], p. 385) :

$$\begin{aligned} \sum_{n=0}^\infty (R_n)^{-1} P_n^{\alpha,\beta}(x) P_n^{\alpha,\beta}(y) r^n &= \frac{1-r}{(1+r)^a} \sum_{m,n\geq 0} \frac{\left(\frac{a}{2}\right)_{m+n} \left(\frac{a+1}{2}\right)_{m+n}}{(\alpha+1)_m (\beta+1)_n} \frac{u^m v^n}{m!n!} \\ &= \frac{1-r}{(1+r)^a} F_4(\frac{a}{2}, \frac{a+1}{2}, \alpha+1, \beta+1; u, v) \end{aligned} \tag{53}$$

where $|r| < 1$, $a = \alpha + \beta + 2$, F_4 is the Appell function and

$$u = \frac{(1-x)(1-y)r}{(1+r)^2} \qquad v = \frac{(1+x)(1+y)r}{(1+r)^2}.$$

The integral representation of F_4 (see [**50**], p 51) yields :

$$q_t(x,y) = \frac{W(y)}{\Gamma(a)} \frac{1-r}{(1+r)^a} \int_0^\infty s^{a-1} e^{-s} {}_0F_1(\alpha+1; \frac{u}{4}s^2) {}_0F_1(\beta+1; \frac{v}{4}s^2) ds$$

Now, from a property of the function ${}_0F_1$ (see [**87**], p 214)

$${}_0F_1(c; w(1-r)/2) {}_0F_1(d; w(1+r)/2) = \sum_{n \geq 0} \frac{P_n^{\alpha,\beta}(r)}{(c)_n (d)_n} w^n, \; \alpha = c-1, \; \beta = d-1,$$

one gets :

$$q_t(x,y) = \frac{W(y)}{\Gamma(a)} \frac{1-r}{(1+r)^a} \int_0^\infty s^{a-1} e^{-s} \sum_{n \geq 0} \frac{P_n^{\alpha,\beta}(z)}{(\alpha+1)_n (\beta+1)_n} A^n s^{2n} \, ds$$

where we set

$$z = \frac{x+y}{1+xy}, \qquad A = \frac{(1+xy)r}{2(1+r)^2}.$$

Applying Fubini's Theorem gives :

$$q_t(x,y) = W(y) \frac{1-r}{(1+r)^a} \sum_{n \geq 0} \frac{(a)_{2n}}{(\alpha+1)_n (\beta+1)_n} P_n^{\alpha,\beta}(z) A^n$$

Letting $r = e^{-t}$, then

$$q_t(x,y) = \frac{W(y) e^{\frac{a-1}{2}t}}{2^{a-1}} \frac{\sinh(t/2)}{(\cosh(t/2))^a} \sum_{n \geq 0} \frac{(a)_{2n}}{(\alpha+1)_n (\beta+1)_n} P_n^{\alpha,\beta}(z) \left[\frac{(1+xy)}{8 \cosh^2(t/2)} \right]^n$$

$$= \frac{W(y) \tanh(t/2) e^{\frac{a-1}{2}t}}{2^{a-1}} \sum_{n \geq 0} \frac{(a)_{2n}}{(\alpha+1)_n (\beta+1)_n} P_n^{\alpha,\beta}(z) \left[\frac{(1+xy)}{8} \right]^n \left(\frac{1}{\cosh(t/2)} \right)^{2n+a-1}.$$

Besides, from (52)

$$q_t(x,y) = \frac{t\, e^{\gamma t}}{2\sqrt{\pi}} \int_0^\infty p_s(x,y)\, s^{-3/2} e^{-\gamma^2 s} \, e^{-\frac{t^2}{4s}} ds = \frac{t\, e^{\gamma t}}{2\sqrt{2\pi}} \int_0^\infty p_{2/r}(x,y)\, r^{-1/2} e^{-2\gamma^2/r} \, e^{-\frac{t^2}{8}r} dr$$

Thus, noting that $\gamma = (a-1)/2$, we get :

$$\int_0^\infty p_{2/r}(x,y)\, r^{-1/2} e^{-2\gamma^2/r} \, e^{-\frac{t^2}{8}r} dr = \frac{\sqrt{2\pi}\, W(y)}{2^{a-1}} \frac{\tanh(t/2)}{t/2}$$
$$\sum_{n \geq 0} \frac{(a)_{2n}}{(\alpha+1)_n (\beta+1)_n} P_n^{\alpha,\beta}(z) \left[\frac{(1+xy)}{8} \right]^n \left(\frac{1}{\cosh(t/2)} \right)^{2n+a-1}.$$

With regard to the integrand, one easily see that the RHS is the Laplace transform of $p_{2/r}(x,y) r^{-1/2} e^{-2\gamma^2/r}$.

1.3. The Jacobi semi-group. Let α, β satisfy $\alpha+\beta+1>0$. The following results are due to Biane, Pitman and Yor (see [**16**], [**96**]) :

$$\int_0^\infty e^{-\frac{t^2}{8}s} f_{C_h}(s)\,ds = \left(\frac{1}{\cosh(t/2)}\right)^h, \qquad h>0 \tag{54}$$

$$\int_0^\infty e^{-\frac{t^2}{8}s} f_{T_h}(s)\,ds = \left(\frac{\tanh(t/2)}{(t/2)}\right)^h, \qquad h>0 \tag{55}$$

where (C_h) and (T_h) are two families of Lévy processes with respective density functions f_{C_h} and f_{T_h} for fixed $h>0$. The densities of C_h and T_1 are given by ([**16**]) :

$$\begin{aligned} f_{C_h}(s) &= \frac{2^h}{\Gamma(h)} \sum_{p\geq 0} (-1)^p \frac{\Gamma(p+h)}{p!} f_{\tau(2p+h)}(s) \\ f_{T_1}(s) &= \sum_{k\geq 0} e^{-\frac{\pi^2}{2}(k+\frac{1}{2})^2 s} \mathbf{1}_{\{s>0\}} \end{aligned}$$

where $\tau(c) = \inf\{r>0; B_r = c\}$ is the Lévy subordinator (the first hitting time of a standard Brownian motion B) with corresponding density :

$$f_{\tau(2p+h)}(s) = \frac{(2p+h)}{\sqrt{2\pi s^3}} \exp\left(-\frac{(2p+h)^2}{2s}\right) \mathbf{1}_{\{s>0\}}.$$

Thus :

$$p_{2/r}(x,y) = \frac{\sqrt{2\pi r} W(y) e^{2\gamma^2/r}}{2^{a-1}} \sum_{n\geq 0} \frac{(a)_{2n}}{(\alpha+1)_n(\beta+1)_n} P_n^{\alpha,\beta}(z) \left[\frac{(1+xy)}{8}\right]^n \times \left(f_{T_1} \star f_{C_{2n+a-1}}\right)(r)$$

or equivalently (where B stands for the Beta function) :

$$p_t(x,y) = \frac{\sqrt{\pi} W(y)}{2^{\alpha+\beta}} \frac{e^{\gamma^2 t}}{\sqrt{t}} \sum_{n\geq 0} \frac{(a)_{2n}}{(\alpha+1)_n(\beta+1)_n} P_n^{\alpha,\beta}(z) \left[\frac{(1+xy)}{8}\right]^n \left(f_{T_1} \star f_{C_{2n+\alpha+\beta+1}}\right)(\frac{2}{t}).$$

1.4. The ultraspherical case. This case corresponds to $\alpha = \beta > -\dfrac{1}{2}$ and we will proceed slight differently. Indeed, $a = 2\alpha+2$ and

$$\begin{aligned} (53) &= \frac{1-r}{(1+r)^{2\alpha+2}} F_4(\alpha+1, \alpha+3/2, \alpha+1, \alpha+1; u, v) \\ &= \frac{1-r}{(1+r)^{2\alpha+2}} \frac{1}{(1-u-v)^{\alpha+3/2}} {}_2F_1(\frac{2\alpha+3}{4}, \frac{2\alpha+5}{4}, \alpha+1; \frac{4uv}{(1-u-v)^2}) \end{aligned}$$

where the last equality follows from (see [**21**])

$$F_4(b,c,b,b;u,v) = (1-u-v)^{-c} {}_2F_1(c/2, (c+1)/2, b; \frac{4uv}{(1-u-v)^2}).$$

Hence,

$$q_t(x,y) = \frac{W(y)e^{\frac{2\alpha+1}{2}t}}{2^{\alpha+1/2}} \frac{\sinh(t)}{(\cosh t - xy)^{\alpha+3/2}} {}_2F_1\left(\frac{2\alpha+3}{4}, \frac{2\alpha+5}{4}, \alpha+1; \frac{(1-x^2)(1-y^2)}{(\cosh t - xy)^2}\right)$$
$$= \frac{W(y)e^{\frac{2\alpha+1}{2}t}}{2^{\alpha+1/2}} \sinh(t) \sum_{n\geq 0} \frac{[(2\alpha+3)/4]_n[(2\alpha+5)/4]_n}{(\alpha+1)_n} \frac{[(1-x^2)(1-y^2)]^n}{(\cosh t - xy)^{2n+\alpha+3/2}}.$$

Besides, for $h > 0$, we may write :

$$\left(\frac{1}{\cosh t - xy}\right)^h = \sum_{k\geq 0} \frac{(h)_k}{k!} \frac{(xy)^k}{(\cosh t)^{k+h}}$$

since $\left|\frac{xy}{\cosh t}\right| < 1 \quad \forall x, y \in]-1,1[, \, \forall t \geq 0$ and where we used :

$$\frac{1}{(1-r)^h} = \sum_{k\geq 0} \frac{(h)_k}{k!} r^k \qquad h > 0, \, |r| < 1.$$

Consequently, using Gauss duplication formula,

$$q_t(x,y) = K_\alpha W(y) e^{\frac{2\alpha+1}{2}t} \tanh(t)$$
$$\sum_{n,k\geq 0} \frac{\Gamma(\nu(n,k,\alpha)+1)(xy)^k}{k!\, n!\, \Gamma(\alpha+n+1)} \left[\frac{(1-x^2)(1-y^2)}{4}\right]^n \left(\frac{1}{\cosh t}\right)^{\nu(n,k,\alpha)}$$

where

$$\nu(n,k,\alpha) = 2n + k + \alpha + 1/2, \qquad K_\alpha = \Gamma(\alpha+1)/[2^{\alpha+1/2}\Gamma(\alpha+3/2)].$$

Thus, since $\gamma = \alpha + 1/2$ when $\alpha = \beta$, one has :

$$\int_0^\infty p_s(x,y)\, s^{-3/2} e^{-\gamma^2 s}\, e^{-\frac{t^2}{4s}} ds = \frac{\sqrt{2\pi}\Gamma(\alpha+1)}{2^\alpha \Gamma(\alpha+3/2)} \frac{\tanh(t)}{t} W(y)$$
$$\sum_{n,k\geq 0} \frac{\Gamma(\nu(n,k,\alpha)+1)(xy)^k}{k!\, n!\, \Gamma(\alpha+n+1)} \left[\frac{(1-x^2)(1-y^2)}{4}\right]^n \left(\frac{1}{\cosh t}\right)^{\nu(n,k,\alpha)}$$

or equivalently :

$$\int_0^\infty p_{1/2s}(x,y) e^{-\frac{\gamma^2}{2s}} e^{-\frac{t^2}{2}s} \frac{ds}{\sqrt{s}} = \frac{\sqrt{\pi}\Gamma(\alpha+1)}{2^\alpha \Gamma(\alpha+3/2)} \frac{\tanh(t)}{t} W(y)$$
$$\sum_{n,k\geq 0} \frac{\Gamma(\nu(n,k,\alpha)+1)(xy)^k}{k!\, n!\, \Gamma(\alpha+n+1)} \left[\frac{(1-x^2)(1-y^2)}{4}\right]^n \left(\frac{1}{\cosh t}\right)^{\nu(n,k,\alpha)}$$

Using (54), (55), f_{C_h} et f_{T_1} (we take $t^2/2$ instead of $t^2/8$), the density is written :

$$p_{1/2s}(x,y) = \frac{\sqrt{\pi s}\Gamma(\alpha+1)}{2^\alpha\Gamma(\alpha+3/2)}W(y)e^{\frac{\gamma^2}{2s}}$$
$$\sum_{n,k\geq 0}\frac{\Gamma(\nu(n,k,\alpha)+1)}{k!n!\Gamma(\alpha+n+1)}\left(\frac{xy}{2}\right)^k\left[\frac{(1-x^2)(1-y^2)}{4}\right]^n f_{T_1}\star f_{C_{\nu(n,k,\alpha)}}(s)$$

Finally

$$(56)\quad p_t(x,y) = \sqrt{\pi}K_\alpha\frac{e^{\gamma^2 t}}{\sqrt{t}}W(y)\sum_{n,k\geq 0}\frac{\Gamma(\nu(n,k,\alpha)+1)(xy)^k}{k!n!\Gamma(\alpha+n+1)}$$
$$\left[\frac{(1-x^2)(1-y^2)}{4}\right]^n f_{T_1}\star f_{C_{\nu(n,k,\alpha)}}(\frac{1}{2t})$$

2. Application to statistics for diffusions processes

2.1. Some properties of the Jacobi process. Usually in probability theory, the Jacobi process is defined on $[-1,1]$ as the unique strong solution of the SDE :

$$dY_t = \sqrt{1-Y_t^2}\,dW_t + (bY_t+c)dt.$$

It is straightforward that $(Y_t)_{t\geq 0} \overset{\mathscr{L}}{=} (X_{t/2})_{t\geq 0}$ where X is the Jacobi process already defined in section 1 with $p = 2b$, $q = 2c$. Using the variable change $y \mapsto (y+1)/2$, the equation above transforms to $(t \to 4t)$:

$$\begin{aligned} dJ_t &= 2\sqrt{J_t(1-J_t)}dW_t + [2(c-b) + 4bJ_t]\,dt \\ &= 2\sqrt{J_t(1-J_t)}dW_t + [d - (d+d')J_t]\,dt \end{aligned}$$

where $d = 2(c-b) = q-p = 2(\beta+1)$ and $d' = -2(c+b) = -(p+q) = 2(\alpha+1)$, which is the Jacobi process of parameters (d,d') already considered by Warren and Yor ([**112**]). Moreover, authors provide the following skew-product : let Z_1, Z_2 be two independent Bessel processes of dimensions d, d' and starting from z, z' respectively. Then :

$$\left(\frac{Z_1^2(t)}{Z_1^2(t)+Z_2^2(t)}\right)_{t\geq 0} \overset{\mathscr{L}}{=} (J_{A_t})_{t\geq 0}\,,\quad A_t := \int_0^t \frac{ds}{Z_1^2(s)+Z_2^2(s)},\quad J_0 = \frac{z}{z+z'}.$$

Using well known properties of squared Bessel processes (see [**101**]), one deduce that if $d \geq 2\,(\beta \geq 0)$ and $z > 0$, then $J_t > 0$ almost surely for all $t > 0$. Since $1-J$ is still a Jacobi process of parameters (d', d), then for $d' \geq 2$, $(\alpha \geq 0)$ and $z' > 0$, $J_t < 1$ almost surely for all $t > 0$. These results fit in the one dimensional case those established in [**?**] for the matrix Jacobi process (Theorem 3.3.2, p.36). Since 0 is a reflecting boundary for Z_1, Z_2 when $0 < d,\, d' < 2\,(-1 < \alpha, \beta < 0)$, then both 0 and 1 are reflecting boundaries for J.

2.2. LDP in the ultraspherical case. Let us consider the following SDE corresponding to the ultraspherical Jacobi process :

$$\begin{cases} dY_t = \sqrt{1-Y_t^2}dW_t + bY_t dt \\ Y_0 = y_0 \in]-1,1[\end{cases} \tag{57}$$

Let $Q^b_{y_0}$ be the law of $(Y_t,\, t \geq 0)$ on the canonical filtered probability space $(\Omega, (\mathcal{F}_t), \mathcal{F})$ where Ω is the space of $]-1,1[$–valued functions. The parameter b is such that $b \leq -1$ (or $\alpha \geq 0$), so that $-1 < Y_t < 1$ for all $t > 0$. The maximum likelihood estimate of b based on the observation of a single trajectory $(Y_s,\, 0 \leq s \leq t)$ under Q^b_0 (see Overbeck [**93**] for more details) is given by

$$\hat{b}_t = \frac{\displaystyle\int_0^t \frac{Y_s}{1-Y_s^2}\, dY_s}{\displaystyle\int_0^t \frac{Y_s^2}{1-Y_s^2}\, ds}. \tag{58}$$

The main result of this section is the following Theorem.

THEOREM 2.1. *When $b \leq -1$, the family $\{\hat{b}_t\}_t$ satisfies a LDP with speed t and good rate function*

$$J_b(x) = \begin{cases} -\dfrac{1}{4}\dfrac{(x-b)^2}{x+1} \quad if \quad x \leq x_0 \\ x + 2 + \sqrt{(b-x)^2 + 4(x+1)} \quad if \quad x > x_0 > b \end{cases} \tag{59}$$

where x_0 is the unique solution of the equation $(b-x)^2 = 4x(x+1) = 0$, $x < -1$.

Proof of Theorem 2.1 : we follow the scheme of Theorem 3.1 in [**117**]. Set :

$$S_{t,x} := \int_0^t \frac{Y_s}{1-Y_s^2}\, dY_s - x \int_0^t \frac{Y_s^2}{1-Y_s^2}\, ds,$$

so that for $x > b$ (resp. $x < b$), $P(\hat{b}_t \geq x) = P(S_{t,x} \geq 0)$ (resp. $P(\hat{b}_t \leq x) = P(S_{t,x} \leq 0)$). Therefore, to derive a large deviation principle on $\{\hat{b}_t\}$, we seek a LDP result for $S_{t,x}/t$ at 0. Let us compute the normalized cumulant generating function $\Lambda_{t,x}$ of $S_{t,x}$:

$$\Lambda_{t,x}(\phi) = \frac{1}{t} \log Q^b_0(e^{\phi S_{t,x}}) \tag{60}$$

From Girsanov formula, the generalized densities are given by

$$\frac{dQ^b_0}{dQ^{b'}_0}\Big|_{\mathcal{F}_t} = \exp\left\{(b-b')\int_0^t \frac{Y_s}{1-Y_s^2}\, dY_s - \frac{1}{2}(b^2 - b'^2)\int_0^t \frac{Y_s^2}{1-Y_s^2}\, ds\right\} \tag{61}$$

Let

$$F(Y_t) = -\frac{1}{2}\log(1-Y_t^2)\,.$$

From Itô formula,

$$F(Y_t) = \int_0^t \frac{Y_s}{1-Y_s^2}\,dY_s + \frac{1}{2}\int_0^t \frac{1+Y_s^2}{1-Y_s^2}\,ds = \int_0^t \frac{Y_s}{1-Y_s^2}\,dY_s + \int_0^t \frac{1}{1-Y_s^2}\,ds - \frac{t}{2}.$$

Let us denote by

$$\mathcal{D}_1(x) = \{\phi\,:\,(b+1)^2 + 2\phi(x+1) \geq 0\}\,.$$

For any $\phi \in \mathcal{D}_1(x)$, we can define $b(\phi,x) = -1 - \sqrt{(b+1)^2 + 2\phi(x+1)}$. With the change of probability defined by (61) taking $b' = b(\phi,x)$, the stochastic integrals simplify to (see [**117**] p. 125 for the details) :

$$\Lambda_t(\phi,x) = \frac{1}{t}\log Q_0^{b(\phi,x)}(\exp(\{\phi + b - b(\phi,x))[F(Y_t) - t/2]\}) \tag{62}$$

When starting from $y_0 = 0$, (56) reads ($t \to t/2$) :

$$\tilde{p}_t(0,y) = \sqrt{2\pi}K_\alpha \frac{e^{\gamma^2 t/2}}{\sqrt{t}} \sum_{n\geq 0} \frac{\Gamma(2n+\alpha+\frac{3}{2})}{4^n n!\Gamma(n+\alpha+1)}(1-y^2)^{n+\alpha} f_{T_1} \star f_{C_{2n+\gamma}}(1/t),$$

where $p = -2(\alpha+1) = 2b \leq -2$ and $\gamma = -(p+1)/2 = \alpha + 1/2$. Denote by

$$\mathcal{D}(x) = \{\phi \in \mathcal{D}_1(x) : G(\phi,x) = b + b(\phi,x) + \phi < 0\}\,. \tag{63}$$

For any $\phi \in \mathcal{D}(x)$, the expectation (62) is finite and a simple computation gives :

$$\begin{aligned}\Lambda_t(\phi,x) &= -\frac{\phi + b - b(\phi,x)}{2} + \frac{1}{t}\log Q_0^{b(\phi,x)}((1-Y_t^2)^{-(\phi+b-b(\phi,x))/2}) \\ &= \Lambda(\phi,x) + \frac{1}{t}\log\frac{\sqrt{2\pi}K_{\alpha(\phi,x)}R_t(\phi,x)}{\sqrt{t}}\end{aligned}$$

where

$$\begin{aligned}R_t(\phi,x) &= \sum_{n\geq 0}\frac{\Gamma(2n - b(\phi,x) + 1/2)}{4^n n!\Gamma(n - b(\phi,x))} B\left(n - \frac{\phi + b + b(\phi,x)}{2}, \frac{1}{2}\right) e^{\gamma^2 t/2} f_{T_1} \star f_{C_{2n+\gamma}}(\frac{1}{t})\,, \\ \alpha(\phi,x) &= -b(\phi,x) - 1\end{aligned}$$

and B stands for the Beta function. With regard to (50), one has for $\phi \in \mathcal{D}(x)$:

$$\lim_{t\to\infty} Q_0^{b(\phi,x)}((1-Y_t^2)^{-(\phi+b-b(\phi,x))/2}) = C_{b,\phi,x}\int_{-1}^1 (1-y^2)^{-[\phi+b+b(\phi,x)]/2-1}\,dy\ < \infty$$

by dominated convergence Theorem. Hence $\Lambda_t \to \Lambda$ as $t \to \infty$. The following lemma, which proof is postponed to the appendix, details the domain $\mathcal{D}(x)$ (see (63)) of Λ_t :

Lemma 2.1. *Denote by*

$$\phi_0(x) = -\frac{(b+1)^2}{2(x+1)}\,.$$

i)<u>If $x < (b^2+3)/2(b-1)$</u> : then $\mathcal{D} = (-\infty, \phi_0(x))$.

ii) <u>*If* $(b^2+3)/2(b-1) < x < -1$</u> *: then* $\mathcal{D}(x) = (-\infty, \phi_1(x))$ *where* $\phi_1(x)$ *is solution of* $G(\phi, x) = 0$.

iii) <u>*If* $x > -1$</u> *: then* $\mathcal{D}(x) = (\phi_0(x), \phi_1(x))$.

In case i) of Lemma above, Λ is steep, i.e. its gradient is infinite at the boundary of the domain (for a precise definition, see [**36**]). It achieves its unique minimum in $\phi_m(x)$ solution of

$$\frac{\partial \Lambda}{\partial \phi}(\phi, x) = 0\,,$$

i.e. $b(\phi(x), x) = x$. It is easy to see that

$$\phi_m(x) = \frac{x+1}{2} - \frac{(b+1)^2}{2(x+1)} < \phi_0(x)\,.$$

Hence, Gärtner-Ellis Theorem gives for $x < b < (b^2+3)/2(b-1)$,

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t} \log P(\hat{b}_t \le x) &= \lim_{t\to\infty} \frac{1}{t} \log P(S_{t,x} \le 0) = \inf_{\phi \le \phi_0(x)} \Lambda(\phi, x) \\ &= \Lambda(\phi_m(x), x) = -\frac{1}{4}\frac{(x-b)^2}{x+1}.\end{aligned}$$

If $b < x < (b^2+3)/2(b-1)$, notice that $\phi_m(x) > 0$ and

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t} \log P(\hat{b}_t \ge x) &= \lim_{t\to\infty} \frac{1}{t} \log P(S_{t,x} \ge 0) = \inf_{\phi \in (0, \phi_0(x)]} \Lambda(\phi, x) \\ &= \Lambda(\phi_m(x), x) = -\frac{1}{4}\frac{(x-b)^2}{x+1}.\end{aligned}$$

In cases ii) and iii) of Lemma 2.1, Λ is not steep. Nevertheless, if the infimum of Λ is reached in $\overset{\circ}{\mathcal{D}}(x)$, we can follow the scheme of Gartner–Ellis theorem for the change of probability in the infimum bound. This infimum is reached if and only if

$$\frac{\partial \Lambda}{\partial \phi}(\phi_1(x), x) > 0\,, \text{i.e. if } \phi_m(x) < \phi_1(x). \tag{64}$$

In case $x+1 > 0$, we know (see proof of Lemma 2.1) that $\phi_1(x) < \phi_m(x)$. If $x+1 < 0$, we check the sign of $G(\phi_m(x), x)$. We get the following dichotomy : Let x_0 denote the unique solution of $g(x) := 4x(x+1) - (b-x)^2 = 0,\ x < -1$. Since g is decreasing on $]-\infty, -1]$ and $g(b^2+3/(2(b-1)) = (3/4)(b+1)^2 > 0 = g(x_0)$, then $x_0 > (b^2+3)/[2(b-1)]$.

• if $(b^2+3)/2(b-1) < x < x_0 < -1$, the derivative $\partial\Lambda/\partial\phi(\phi_1(x), x) > 0$, Λ achieves its minimum on $\phi_m(x)$ and

$$\lim_{t\to\infty} \frac{1}{t} \log P(\hat{b}_t \ge x) = \Lambda(\phi_m(x), x) = -\frac{(x-b)^2}{4(x+1)}\,.$$

• if $x_0 < x < -1$ or $x > -1$, then $\partial\Lambda/\partial\phi(\phi_1(x),x) < 0$. We apply Theorem 4.1 of the appendix, which is due to Zani [**117**]. Let us verify that the assumptions are satisfied, and more precisely that Λ_t can take form (67). Indeed, the only singularity $\phi_1(x)$ of R_t comes from $B\left(n-[\phi+b+b(\phi,x)]/2, 1/2\right)$ when $n=0$, and more precisely, from $\Gamma(-[\phi+b+b(\phi,x)]/2)$. We can write

$$\Lambda_t(\phi,x) = \Lambda(\phi,x) + \frac{1}{t}\log\Gamma\left(-\frac{\phi+b+b(\phi,x)}{2}\right) + \frac{1}{t}\log\frac{\sqrt{2\pi}K_{\alpha(\phi,x)}\tilde{R}_t(\phi,x)}{\sqrt{t}}\,, \tag{65}$$

where

$$\tilde{R}_t(\phi,x) = R_t(\phi,x)/\Gamma(-[\phi+b+b(\phi,x)]/2). \tag{66}$$

Now

$$\forall n \geq 0, \quad B\left(n-\frac{\phi+b+b(\phi,x)}{2},\frac{1}{2}\right)/\Gamma\left(-\frac{\phi+b+b(\phi,x)}{2}\right)$$

is analytic on some neighbourhood of $\phi_1(x)$. Besides,

$$\lim_{\phi\to\phi_1(x)\,,\phi<\phi_1(x)} \frac{b+\phi+b(\phi,x)}{\phi-\phi_1(x)} = c > 0\,,$$

and since $\lim_{\rho\to 0^+}\rho\Gamma(\rho) = 1$, then $\phi_1(x)$ is a pole of order one of $\Gamma\left(\phi+b+b(\phi,x)/2\right)$ and one writes :

$$\frac{1}{t}\log\Gamma\left(-\frac{\phi+b+b(\phi,x)}{2}\right) = -\frac{\log(\phi_1(x)-\phi)}{t} + \frac{h(\phi)}{t}\,.$$

The function h is analytic on $\mathcal{D}(x)$ and can be extended to an analytic function on $]\phi_1(x)-\xi,\phi_1(x)+\xi[$ for some positive ξ.
Finally, to satisfy **Assumption 1** of the appendix, we focus on $\tilde{R}_t(\phi,x)/\sqrt{t}$ and show that it is bounded uniformly as $t\to\infty$. To proceed, we shall prove that this ratio is bounded from above and below away from 0. Setting $A_n(t) := e^{\gamma^2 t/2} f_{T_1}\star f_{C_{2n+\gamma}}(1/t)$, one has :

$$\begin{aligned}
\frac{A_n(t)}{\sqrt{t}} &\leq \frac{e^{\gamma^2 t/2}}{\sqrt{t}}\sum_{k,l\geq 0} U_{k,n}\int_0^{1/t}\exp -\frac{1}{2}\left[\frac{(2n+2k+\gamma)^2}{s} + \pi^2(l+\frac{1}{2})^2(\frac{1}{t}-s)\right]\frac{ds}{s^{3/2}} \\
&= \frac{e^{\gamma^2 t/2}}{\sqrt{t}}\sum_{k,l\geq 0} U_{k,n}\int_t^{\infty}\exp -\frac{1}{2}\left[(2n+2k+\gamma)^2 s + \pi^2(l+\frac{1}{2})^2(\frac{s-t}{ts})\right]\frac{ds}{\sqrt{s}} \\
&< e^{-2n^2}\sum_{k,l\geq 0} U_{k,n}e^{-2k^2}\int_t^{\infty}\exp -\frac{1}{2}\left[(2n+2k+\gamma)^2(s-t) + \pi^2(l+\frac{1}{2})^2(\frac{s-t}{ts})\right]\frac{ds}{\sqrt{ts}} \\
&= e^{-2n^2}\sum_{k,l\geq 0} U_{k,n}e^{-2k^2}\int_0^{\infty}\exp -\frac{1}{2}\left[(2n+2k+\gamma)^2 s + \pi^2 l^2(\frac{s}{t(t+s)})\right]\frac{ds}{\sqrt{t(t+s)}}
\end{aligned}$$

with

$$U_{k,n} = \frac{\Gamma(2n+k+\gamma)2^{2n+\gamma}(2n+2k+\gamma)}{k!\Gamma(2n+\gamma)}.$$

Let $\Theta(x) = \sum_{l\in\mathbb{Z}} e^{-\pi l^2 x} = 1 + 2\sum_{l\geq 1} e^{-\pi l^2 x}$ denote the Jacobi Theta function ([**16**]). Then

$$\frac{A_n(t)}{\sqrt{t}} < e^{-2n^2}\left[\sum_{k\geq 0} U_{k,n}e^{-2k^2}\int_0^\infty \exp -\left[\frac{(2n+2k+\gamma)^2 s}{2}\right]\Theta\left(\frac{\pi s}{2t(t+s)}\right)\frac{ds}{\sqrt{t(t+s)}}\right]$$
$$+ e^{-2n^2}C(n,t)$$

where

$$C(n,t) = \frac{1}{2\sqrt{t}}\sum_{k,l\geq 0} U_{k,n}e^{-2k^2}\int_0^\infty \exp -\left[\frac{(2n+2k+\gamma)^2 s}{2}\right]\frac{ds}{\sqrt{t+s}}$$

Recall that $\Theta(x) = (1/\sqrt{x})\Theta(1/x)$ ([**16**]), which yields :

$$\frac{A_n(t)}{\sqrt{t}} < e^{-2n^2}\sum_{k\geq 0} U_{k,n}e^{-2k^2}\int_0^\infty \exp -\left[\frac{(2n+2k+\gamma)^2 s}{2}\right]\Theta\left(\frac{2t(t+s)}{\pi s}\right)\frac{ds}{\sqrt{s}}+\frac{C(n)}{2\sqrt{t}}$$

where

$$C(n) = e^{-2n^2}\sum_{k,l\geq 0} U_{k,n}e^{-2k^2}\int_0^\infty \exp -\left[\frac{(2n+2k+\gamma)^2 s}{2}\right]\frac{ds}{\sqrt{s}}$$

Since $e^{-l^2 z} < e^{-lz}$, then $\Theta(z) \leq 3$ for $z > 1$. Hence, as $2t/\pi \leq 2t(t+s)/(\pi s)$, then for t large enough :

$$\frac{A_n(t)}{\sqrt{t}} < 3e^{-2n^2}\sum_{k\geq 0} U_{k,n}e^{-2k^2}\int_0^\infty \exp -\left[\frac{(2n+2k+\gamma)^2 s}{2}\right]\frac{ds}{\sqrt{s}} + C(n) < 4C(n).$$

The upper bound follows since $\sum_n C(n) < \infty$. Besides,

$$\frac{\tilde{R}_t(\phi,x)}{\sqrt{t}} > \frac{\sqrt{\pi}\Gamma(1/2-b(\phi,x))}{\Gamma(-b(\phi,x))\Gamma\{[1-(\phi+b+b(\phi,x)]/2\}}\frac{A_0(t)}{\sqrt{t}}$$
$$= C(b,\phi,x)\sum_{k,l\geq 0}(-1)^k V_k\int_0^\infty \exp -\frac{1}{2}\left[(2k+\gamma)^2 s + \pi^2(l+\frac{1}{2})^2(\frac{s}{t(t+s)})\right]\frac{ds}{\sqrt{t(t+s)}}$$

where $V_k(t) := U_{k,0}e^{-2k(k+\gamma)t}$. One may choose t large enough independent of k such that $V_k(t) \geq V_{k+1}(t)$ for all $k \geq 0$. In fact, such t satisfies :

$$e^{2(2k+\gamma+1)t} \geq e^{2t} \geq \sup_{k\geq 0}\frac{U_{k+1,0}}{U_{k,0}} = \sup_{k\geq 0}\frac{(k+\gamma)(2k+\gamma+2)}{(k+1)(2k+\gamma)}$$

Then :

$$\frac{\tilde{R}_t}{\sqrt{t}} > C(b,\phi,x)[V_0(t)-V_1(t)]\sum_{l\geq 0}\int_0^\infty \exp -\frac{1}{2}\left[\gamma^2 s+\pi^2(l+\frac{1}{2})^2(\frac{s}{t(t+s)})\right]\frac{ds}{\sqrt{t(t+s)}}$$
$$> C(b,\phi,x)[\gamma 2^\gamma - V_1(t)]\sum_{l\geq 0}\int_0^\infty \exp -\frac{1}{2}\left[\gamma^2 s+\pi^2(l+1)^2(\frac{s}{t(t+s)})\right]\frac{ds}{\sqrt{t(t+s)}}$$
$$= \frac{C(b,\phi,x)}{2}[\gamma 2^\gamma - V_1(t)]\left\{\int_0^\infty e^{-\gamma^2 s/2}\Theta\left(\frac{\pi s}{2t(t+s)}\right)\frac{ds}{\sqrt{t(t+s)}} - C(t)\right\}.$$

where

$$C(t) = \frac{1}{2\sqrt{t}}\int_0^\infty e^{-\gamma^2 s/2}\frac{ds}{\sqrt{(t+s)}} < c\int_0^\infty e^{-\gamma^2 s/2}\frac{ds}{\sqrt{s}}, \quad c < \sqrt{\frac{2}{\pi}}.$$

for t large enough. Following the same scheme as for the upper bound, one gets :

$$\frac{\tilde{R}_t}{\sqrt{t}} > \frac{C(b,\phi,x)}{2}\gamma 2^\gamma\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\gamma^2 s/2}\Theta\left(\frac{2t(t+s)}{\pi s}\right)\frac{ds}{\sqrt{s}} - C(t)\right\}$$
$$> \frac{C(b,\phi,x)}{2}\gamma 2^\gamma\left(\sqrt{\frac{2}{\pi}} - c\right)\int_0^\infty e^{-\gamma^2 s/2}\frac{ds}{\sqrt{s}} > 0. \qquad \square$$

As a result,

$$\lim_{t\to\infty}\frac{1}{t}\log P(\hat{b}_t \geq x) = \Lambda(\phi_1(x),x) = -(x+2+\sqrt{(b-x)^2+4(x+1)}),$$

which ends the proof of Theorem 2.1. ■

2.3. Jacobi-squared Bessel processes duality. By Itô's formula and Lévy criterion, one claims that $(Y_t^2)_{t\geq 0}$ is a $[0,1]$-valued Jacobi process of parameters $d=1$, $d'=-2b\geq 2$. Indeed :

$$dZ_t := d(Y_t^2) = 2Y_t dY_t + \langle Y\rangle_t = 2Y_t\sqrt{1-Y_t^2}dW_t + [(2b-1)Y_t^2+1]dt$$
$$= 2\sqrt{Z_t(1-Z_t)}\text{sgn}(Y_t)dW_t + [(2b-1)Z_t+1]dt$$
$$= 2\sqrt{Z_t(1-Z_t)}dB_t + [(2b-1)Z_t+1]dt$$

Using the skew product previously stated, there exists R, a squared Bessel process of dimension $d' = 2(\nu+1) = -2b$ and starting from 0 so that :

$$\hat{\nu}_t := -\hat{b}_t - 1 = \frac{\log(1-Y_t^2)+t}{2\int_0^t \frac{Y_s^2}{1-Y_s^2}ds}$$

is another estimator of ν based on a Jacobi trajectory observed till time t. Set $t = \log u$, then

$$\hat{\nu}_{\log u} = \frac{\log[u(1 - Y^2_{\log u})]}{2\int_0^{\log u} \frac{Y_s^2}{1-Y_s^2} ds} = \frac{\log[u(1 - Y^2_{\log u})]}{2\int_1^{u} \frac{Y^2_{\log s}}{s(1-Y^2_{\log s})} ds}$$

and $\{\hat{\nu}_{\log u}\}_u$ satisfies a LDP with speed $\log u$ and rate function $J_{-(\nu+1)}(-(x+1))$.

When starting at $R_0 = 1$, the MLE of ν based on a Bessel trajectory is given by (cf [**117**], p. 132) :

$$\hat{\nu}_t^1 = \frac{\int_0^t \frac{dX_s}{X_s} - 2\int_0^t \frac{ds}{X_s}}{2\int_0^t \frac{ds}{X_s}} = \frac{\log(X_t)}{2\int_0^t \frac{ds}{X_s}}$$

with associated rate function :

$$I_\nu(x) = \begin{cases} \frac{(x-\nu)^2}{4x} & \text{if} \quad x \geq x_1 \quad := \frac{-(\nu+2)+2\sqrt{\nu^2+\nu+1}}{3} \\ 1 - x + \sqrt{(\nu - x)^2 - 4x} & \text{if} \quad x < x_1 \end{cases}$$

A glance at both rate functions gives $I_\nu(x) = J_{-(\nu+1)}(-(x+1))$ and $x_0 = -(x_1+1)$. We end the paper with computing moments of the Jacobi process starting from 0.

3. On the moments of Jacobi process

In this section, we consider a Jacobi process $(J_t)_{t\geq 0}$ defined on $[0,1]$, and starting from $J_0 = 0$ a. s, that is :

$$J_t = \int_0^t 2\sqrt{J_s(1-J_s)}dW_s + \int_0^t (d - (d'+d)J_s)ds.$$

By Itô's formula, one has for $n \geq 1$:

$$\begin{aligned} J_t^n &= \int_0^t nJ_s^{n-1}dJ_s + \frac{1}{2}\int_0^t n(n-1)J_s^{n-2}d\langle J\rangle_s \\ &= M_t + n\int_0^t J_s^{n-1}(d - (d'+d)J_s)ds + 2n(n-1)\int_0^t J_s^{n-2}J_s(1-J_s)ds \\ &= M_t - n(d'+d+2n-2)\int_0^t J_s^n ds + n(d+2n-2)\int_0^t J_s^{n-1}ds \end{aligned}$$

where $M_t = 2n\int_0^t J_s^{n-1}\sqrt{J_s(1-J_s)}dB_s$ is the local martingale part. Since $J_s \in [0,1]$, then $\mathbb{E}(\langle M\rangle_t) \leq t$ and hence, $(M_t)_{t\geq 0}$ is a martingale. Taking the expectation and using Fubini Theorem, we get :

$$\mathbb{E}(J_t^n) = -n(d'+d+2n-2)\int_0^t \mathbb{E}(J_s^n)ds + n(d+2n-2)\int_0^t \mathbb{E}(J_s^{n-1})ds.$$

Since $s \to \mathbb{E}(J_s^n)$ is continuous, then

$$f_n'(t) := \frac{d}{dt}\mathbb{E}(J_t^n) = -n(d'+d+2n-2)f_n(t) + n(d+2n-2)f_{n-1}(t), \qquad f_n(0) = 0.$$

We shall show by induction that :

PROPOSITION 3.1. *For all $n \geq 1$, the Jacobi moments are given by :*

$$f_n(t) == \prod_{i=0}^{n-1}(d+2i) \times \left(\frac{1}{\prod_{i=0}^{n-1}(d'+d+2i)} + \sum_{k=1}^{n} \frac{(-1)^k \binom{n}{k}(d'+d+4k-2)e^{-k(d'+d+2k-2)t}}{\prod_{i=k-1}^{k+n-1}(d'+d+2i)} \right)$$

Proof : we have to show that :

$$f_{n+1}(t) = \prod_{i=0}^{n}(d+2i) \times \left(\frac{1}{\prod_{i=0}^{n}(d'+d+2i)} + \sum_{k=1}^{n+1} \frac{(-1)^k \binom{n+1}{k}(d'+d+4k-2)e^{-k(d'+d+2k-2)t}}{\prod_{i=k-1}^{k+n}(d'+d+2i)} \right).$$

Indeed, the recurrence relation yields :

$$f_{n+1}(t) = u(t)e^{-(n+1)(d'+d+2n)t}$$

which implies that :

$$u'(t) = (n+1)(d+2n)e^{(n+1)(d'+d+2n)t}f_n(t) =$$

$$\prod_{i=0}^{n}(d+2i) \times \left(\frac{(n+1)e^{(n+1)(d'+d+2n)t}}{\prod_{i=0}^{n-1}(d'+d+2i)} + \sum_{k=1}^{n} \frac{(-1)^k (n+1)!(d'+d+4k-2)e^{(n+1-k)(d'+d+2(n+k))t}}{k!(n-k)!\prod_{i=k-1}^{k+n-1}(d'+d+2i)} \right).$$

Thus, noting that :

$$\begin{aligned}(n+1)(d'+d+2n) - k(d'+d+2k-2) &= (d'+d)(n+1-k) + 2(n-k)(n+k) + 2(n+k) \\ &= (d'+d)(n+1-k) + 2(n-k)(n+1-k) \\ &= (n+1-k)(d'+d+2(n-k)),\end{aligned}$$

we get :

$$u(t) = \prod_{i=0}^{n}(d+2i) \times (\frac{e^{(n+1)(d'+d+2n)t}}{\prod_{i=0}^{n}(d'+d+2i)} + \sum_{k=1}^{n} \frac{(-1)^k \binom{n+1}{k}(d'+d+4k-2)e^{(n+1-k)(d'+d+2(n+k))t}}{\prod_{i=k-1}^{k+n}(d'+d+2i)} + K)$$

where

$$K = -\left(\frac{1}{\prod_{i=0}^{n}(d'+d+2i)} + \sum_{k=1}^{n} \frac{(-1)^k \binom{n+1}{k}(d'+d+4k-2)}{\prod_{i=k-1}^{k+n}(d'+d+2i)} \right).$$

Now, it is very easy to see that :

$$K = \frac{(-1)^{n+1}}{\prod_{i=n}^{2n}(d'+d+2i)}.$$

The two first terms of the sum contribute to :

$$\begin{aligned}\frac{1}{\prod_{i=0}^{n}(d'+d+2i)} - \frac{(n+1)(d'+d+2)}{\prod_{i=0}^{n+1}(d'+d+2i)} &= \frac{d'+d+2n+2-(n+1)(d'+d+2)}{\prod_{i=0}^{n+1}(d'+d+2i)} \\ &= -\frac{n}{\prod_{i=1}^{n+1}(d'+d+2i)}.\end{aligned}$$

Additionning the third term, we obtain :

$$\frac{n(n+1)(d'+d+6)}{2\prod_{i=1}^{n+2}(d'+d+2i)} - \frac{n}{\prod_{i=1}^{n+1}(d'+d+2i)} = \frac{n}{2}\left(\frac{(n+1)(d'+d+6)-2(d'+d+2n+4)}{\prod_{i=1}^{n+2}(d'+d+2i)}\right)$$
$$= \frac{n(n-1)}{2\prod_{i=2}^{n+2}(d'+d+2i)}.$$

We follow in the same way to find the expression above which ends the proof.

4. Appendix

4.1. A large deviations principle in a non steep case. Let $\{Y_t\}_{t\geq 0}$ be a family of real random variables defined on $(\Omega, \mathcal{F}, P)$, and denote by μ_t the distribution of Y_t. Suppose $-\infty < m_t := EY_t < 0$. We look for large deviations bounds for $P(Y_t \geq y)$. Let Λ_t be the n.c.g.f. of Y_t :

$$\Lambda_t(\phi) = \frac{1}{t}\log\, E(\exp\{\phi t Y_t\}),$$

and denote by D_t the domain of Λ_t. We assume that there exists $0 < \phi_1 < \infty$ such that for any t

$$\sup\{\phi : \phi \in D_t\} = \phi_1$$

and $[0, \phi_1) \subset D_t$. We assume also that for $\phi \in D$

Assumption 1.

$$\Lambda_t(\phi) = \Lambda(\phi) - \frac{\alpha}{t}\log(\phi_1 - \phi) + \frac{R_t(\phi)}{t} \tag{67}$$

where
- $\alpha > 0$
- Λ is analytic on $(0, \phi_1)$, convex, with finite limits at endpoints, such that $\Lambda'(0) < 0$, $\Lambda'(\phi_1) < \infty$, and $\Lambda''(\phi_1) > 0$.
- R_t is analytic on $(0, \phi_1)$ and admits an analytic extension on a strip $D_\beta^\gamma = (\phi_1 - \beta, \phi_1 + \beta) \times (-\gamma, \gamma)$, where β and γ are independent of t.
- $R_t(\phi)$ converges as $t \to \infty$ to some $R(\phi)$ uniformly on any compact of D_β^γ.

Theorem 4.1. *Under 1*
For any $\Lambda'(0) < y < \Lambda'(\phi_1)$,

$$\lim_{t\to+\infty} \frac{1}{t}\log\, P(Y_t \geq y) = -\sup_{\phi\in(0,\phi_1)}\{y\phi - \Lambda(\phi)\}. \tag{68}$$

For any $y \geq \Lambda'(\phi_1)$,

$$\lim_{t\to+\infty} \frac{1}{t}\log\, P(Y_t \geq y) = -y\phi_1 + \Lambda(\phi_1). \tag{69}$$

The rate function is continuously differentiable with a linear part.

4.2. Proof of Lemma 2.1 : Note first that $(b^2+3)(2(b-1)) < -1$ if $b < -1$ and that the condition $\phi \in \mathcal{D}_1(x) \Rightarrow \phi \geq \phi_0(x)$ if $x > -1$ and $\phi \leq \phi_0(x)$ if $x < -1$. To examine the behaviour of G, we compute

$$\frac{\partial G}{\partial \phi}(\phi, x) = 1 - \frac{x+1}{\sqrt{(b+1)^2 + 2\phi(x+1)}}\,.$$

• If $x+1 < 0$, $\dfrac{\partial G}{\partial \phi} > 0$ and $G(\cdot, x)$ is increasing. Then we see easily that $G(\phi_0(x), x) < 0$ iff $x < (b^2+3)(2(b-1))$, which determines cases i) and ii).

• If $x+1 > 0$, $\phi \to \dfrac{\partial G}{\partial \phi}$ is increasing hence there exists $\tilde{\phi}(x)$ such that $\dfrac{\partial G}{\partial \phi}(\tilde{\phi}(x), x) = 0$. We compute

$$\tilde{\phi}(x) = \frac{x+1}{2} - \frac{(b+1)^2}{2(x+1)} = \phi_m(x)\,.$$

We see that $G(\tilde{\phi}(x), x) < 0$, and there exists $\phi_1(x) < \tilde{\phi}(x)$ such that $G(\phi_1(x), x) = 0$, and $\mathcal{D}(x) = (\phi_0(x), \phi_1(x))$. $\square$

Remark. This work was firstly motivated by an open question from matrix theory and more precisely while studying the matrix Jacobi process ([**43**]).

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